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Matched Filter Issue

An Introduction to Matched Filters

Joint Estimation of Delay, Doppler, and Doppler Rate

Processing Gain Against Reverberation

On New Classes of Matched Filters

Correlation Detection of Signals Perturbed by a Random Channel

A Matched Filter Communication System for Multipath Systems

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Quaternary Codes for Pulsed Radar

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IRE Transactions on Information Theory

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TABLE OF CONTENTS

PAGE

Introduction — *Paul E. Green, Jr.* 310

Contributions

An Introduction to Matched Filters — *George L. Turin* 311

Joint Estimation of Delay, Doppler, and Doppler Rate — *Phillip Bello* 330

Processing Gains Against Reverberation (Clutter) Using Matched Filters — *E. C. Westerfield, R. H. Prager, and J. L. Stewart* 342

On New Classes of Matched Filters and Generalizations of the Matched Filter Concept — *David Middleton* 349

Correlation Detection of Signals Perturbed by a Random Channel — *Thomas Kailath* 361

A Matched Filter Communication System for Multipath Channels — *Steven M. Sussman* 367

A Matched Filter Detection System for Complicated Doppler Shifted Signals — *Robert M. Lerner* 373

Optical Data Processing and Filtering Systems — *L. J. Cutrona, E. N. Leith, C. J. Palermo, and L. J. Porcello* 386

Quaternary Codes for Pulsed Radar — *George R. Welti* 400

Correspondence

Optimum Approximation to a Matched Filter Response — *J. W. Craig* 409

Coherent Detection by Quasi-Orthogonal Square-Wave Pulse Functions — *R. C. Titsworth* 410

Solution of an Integral Equation Occurring in Multipath Communication Problems — *Thomas Kailath* 412

Correction to "On the Detection of Stochastic Signals in Additive Normal Noise—Part I" — *David Middleton* 412

Contributors 413

Abstracts Section 416

Preface to the Matched Filter Issue

This issue of the PGIT TRANSACTIONS is the first in what is intended to be a series devoted to special topics. Such Special Issues will be organized from time to time whenever the Editors feel that they have identified a sizeable body of new and previously unavailable results having a common unity.

This Special Issue takes as its point of departure a disarmingly humble and specific notion: that the correlation of one waveform with another can be carried out by 1) passing the first waveform through a linear system whose impulse response is the time reverse of the second waveform, and 2) observing the output at a certain instant of time. If the two waveforms are made the same, we say that the filter is "matched" to the input waveform. The filter output as a function of real time is then the autocorrelation function of the waveform.

The subject was introduced in 1943 in North's study of maximization signal-to-noise ratio out of the IF of a pulse radar. Correlation detection was studied at first as a separate subject, but the equivalence of the two operations was soon appreciated. By now they have long since fused and the only meaningful distinction lies in the matter of actual hardware—whether the correlation operations take place using multipliers and integrators, or alternatively by observing the sampled outputs of matched filters. Often, but not always, there are compelling engineering reasons for preferring the latter approach.

No matter what formalism is used to view a given communication or detection situation, Gaussian noise statistics lead usually to some form of correlation or matched filtering as a part of the set of operations that will perform the desired function most efficiently. This appears to be true even when in addition to the noise there are other perturbing factors present, such as randomly varying multipath, uncertainties in signal delay or Doppler shift, Doppler or delay smearing, or unwanted clutter.

The present issue contains a sampling of some important aspects of the by now large subject of "matched filter communication and detection." A central feature of this volume is the tutorial survey by Turin prepared by special invitation. This paper will serve as a valuable introduction and guide to the literature for those who may be reading about matched filter theory for the first time, and should help to unify the various notions for those already familiar with separate aspects of the field.

The next paper, by Bello, treats the question of uncertainties in the simultaneous estimation of range, velocity, and acceleration of targets observed by pulse radar. Westerfield, Prager, and Stewart, analyze the use of a matched filter radar or sonar in discriminating targets from clutter by exploiting the fact that the clutter may have a characteristic distribution of returned power as a function of range and Doppler that is different from that of the signal. Middleton's paper deals with a formulation of the statistical detection problem in such a way that over a wide class of conditions, some sort of filter specification appears as the result. In general these filters may be time-variant, but in the simple white noise case they are the same as the more familiar fixed filters matched in the sense indicated above.

Kailath's paper discusses a point of view in which earlier work on optimum receivers for channels with certain multiplicative disturbances (*e.g.* varying multipath) is shown to make use of a maximum-likelihood estimate of the transmitted waveform as modified by the multiplicative disturbance. Sussman describes an operating antimultipath communication system built around the matched filter notion. A large time-bandwidth product is required in such applications, and in this case the filter is made up of a large number of narrow-band elements spaced uniformly in frequency.

The design of matched filter equipment is also discussed by Lerner, Cutrona, *et al.*, and Craig. Lerner is interested in matched filters having a high degree of time delay discrimination and an ability to accommodate and measure Doppler shifts. The basic matched filter he uses is made up of a number of wideband elements spaced uniformly in delay, *i.e.*, a tapped delay line.

Cutrona, Porcello, Palermo, and Leith, discuss the treatment of signals by optical processors. An interesting feature of this class of techniques is the ease with which large numbers of hypotheses can be tested simultaneously, a capability that appears to be increasingly necessary as communication and detection systems become more complex. Craig analyzes what can be done in designing a linear filter for best performance as a matched filter when only a limited number of design parameters may be varied.

Two specific approaches to the important problem of choosing the waveforms so as to have desirable auto- and cross-correlation properties are presented by Welti and Titsworth. This problem is touched on in Lerner's paper as well, and by implication at least, by Sussman.

The Abstracts Section, a new regular feature of the TRANSACTIONS, appears for the first time. The Editors of this section have prepared a number of abstracts summarizing various important papers relating to matched filters.

The reader will recognize at once that the collection of papers in this issue is more indicative than it is exhaustive. More is being added to the story daily. It is hoped that this Special Issue will stimulate a wider discussion and suggest profitable directions for further work.

PAUL E. GREEN, JR.
Editor of Matched Filter Issu

An Introduction to Matched Filters*

GEORGE L. TURIN†

Summary—**In a tutorial exposition, the following topics are cussed: definition of a matched filter; where matched filters se; properties of matched filters; matched-filter synthesis and nal specification; some forms of matched filters.**

I. Foreword

N this introductory treatment of matched filters, an attempt has been made to provide an engineering insight into such topics as: where these filters arise, ıat their properties are, how they may be synthesized, :. Rigor and detail are purposely avoided, on the theory at they tend, on first contact with a subject, to obscure ndamental concepts rather than clarify them. Thus, r example, although it is not assumed that the reader conversant with statistical estimation and hypothesis-sting theories, the pertinent results of these are invoked thout mathematical proof; instead, they are justified an appeal to intuition, starting with simple cases d working up to greater and greater complexity. Such presentation is admittedly not sufficient for a com-etely thorough understanding: it is merely a prelude. is hoped that the interested reader will fill in the gaps mself by consulting the cited references at his leisure.

Of course, one must always start somewhere, and here is with the assumption that the reader is already miliar with the elements of probability theory and ear filter theory—that is, with such things as prob-ility density functions, spectra, impulse response func-ns, transfer functions, and so forth. If he is not, refer-ce to Chapters 1 and 2 of [57] and Chapter 9 of [7] ll probably suffice.

The bibliography, although lengthy, is not meant to complete, nor could it be. Aside from the inevitable ability of the author to be familiar with the entire classified literature on the subject, there is an extensive dy of classified literature, much of it precedent to the classified literature, which of course could not be cited. the latter, it must be said regretfully, large portions ould not have been classified in the first place, or ould long since have been declassified.

No bibliography can satisfactorily reflect the influence personal conversations with colleagues on an author's ughts about a subject. The present author would ecially like to acknowledge the many he has had er the years with Drs. W. B. Davenport, Jr., R. M. no, P. E. Green, Jr., and R. Price; this, without attrib-ng to them in any way the inadequacies of what ows.

* Manuscript received by the PGIT, January 23, 1960.
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II. Definition of a Matched Filter

If $s(t)$ is any physical waveform, then a filter which is matched to $s(t)$ is, by definition, one with impulse response

$$h(\tau) = ks(\Delta - \tau), \tag{1}$$

where k and Δ are arbitrary constants. In order to envisage the form of $h(t)$, consider Fig. 1, in part (a) of which is shown a wave train, $s(t)$, lasting from t_1 to t_2. By reversing the direction of time in part (a), *i.e.*, letting $\tau = -t$, one obtains the reversed train, $s(-\tau)$, of part (b). If this latter waveform is now delayed by Δ seconds, and its amplitude multiplied by k, the resulting waveform—part (c) of Fig. 1—is the matched-filter impulse response of (1).[1]

The transfer function of a matched filter, which is the Fourier transform of the impulse response, has the form

$$\begin{aligned} H(j2\pi f) &= \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}\,d\tau \\ &= k\int_{-\infty}^{\infty} s(\Delta - \tau)e^{-j2\pi f\tau}\,d\tau \\ &= ke^{-j2\pi f\Delta}\int_{-\infty}^{\infty} s(\tau')e^{j2\pi f\tau'}\,d\tau', \end{aligned} \tag{2}$$

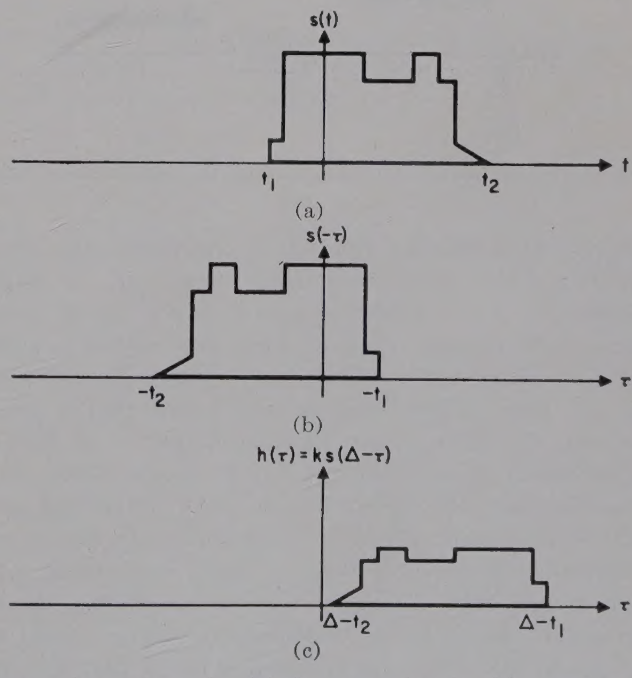


Fig. 1—(a) A wave train; (b) the reversed train; (c) a matched-filter impulse response.

[1] For some types of synthesis of $h(\tau)$—for example, as the impulse response of a passive, linear, electrical network—Δ is constrained by realizability considerations to the region $\Delta \geq t_2$. If $t_2 = \infty$, approximations are sometimes necessary. The problems of realization will be considered more fully later.

where the substitution $\tau' = \Delta - \tau$ has been made in going from the third to the fourth member of (2). Now, the spectrum of $s(t)$, *i.e.*, its Fourier transform, is:[2]

$$S(j2\pi f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}\,dt. \tag{3}$$

Comparison of (2) and (3) reveals, then, that

$$H(j2\pi f) = kS(-j2\pi f)e^{-j2\pi f\Delta} = kS^*(j2\pi f)e^{-j2\pi f\Delta}. \tag{4}$$

That is, except for a possible amplitude and delay factor of the form $ke^{-j2\pi f\Delta}$, the transfer function of a matched filter is the complex conjugate of the spectrum of the signal to which it is matched. For this reason, a matched filter is often called a "conjugate" filter.

Let us postpone further study of the characteristics of matched filters until we have gained enough familiarity with the contexts in which they appear to know what properties are important enough to investigate.

III. Where Matched Filters Arise

A. Mean-Square Criteria

Perhaps the first context in which the matched filter made its appearance [31], [55] is that depicted in Fig. 2.

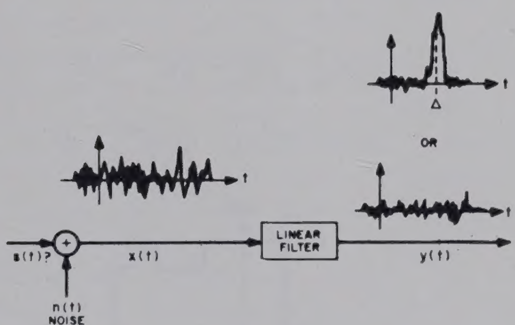


Fig. 2—Pertaining to the maximization of signal-to-noise ratio.

Suppose that one has received a waveform, $x(t)$, which consists either solely of a white noise, $n(t)$, of power density $N_0/2$ watts/cps;[3] *or* of $n(t)$ *plus* a signal, $s(t)$—say a radar return—of known form. One wishes to determine which of these contingencies is true by operating on $x(t)$ with a linear filter in such a way that if $s(t)$ *is* present, the filter output at some time $t = \Delta$ will be considerably greater than if $s(t)$ is absent. Now, since the filter has been assumed to be linear, its output, $y(t)$, will be composed of a noise component $y_n(t)$, due to $n(t)$ only, and, if $s(t)$ is present, a signal component $y_s(t)$, due to $s(t)$ only. A simple way of quantifying the requirement that $y(\Delta)$ be "considerably greater" when $s(t)$ is present than when $s(t)$ is absent is to ask that the filter make the instantaneous power in $y_s(\Delta)$ as large as possibl compared to the average power in $n(t)$ at time Δ.

Assuming that $n(t)$ is stationary, the average power i $n(t)$ at any instant is the integrated power under th noise power density spectrum at the filter output. I $G(j2\pi f)$ is the transfer function of the filter, the outpu noise power density is $(N_0/2)\,|\,G(j2\pi f)\,|^2$; the outpu noise power is therefore

$$\frac{N_0}{2}\int_{-\infty}^{\infty} |\,G(j2\pi f)\,|^2\,df. \tag{5}$$

Further, if $S(j2\pi f)$ is the input signal spectrum, the $S(j2\pi f)G(j2\pi f)$ is the output signal spectrum, and $y_s(\Delta)$ is the inverse Fourier transform of this, evaluated a $t = \Delta$; that is,

$$y_s(\Delta) = \int_{-\infty}^{\infty} S(j2\pi f)G(j2\pi f)e^{j2\pi f\Delta}\,df. \tag{6}$$

The ratio of the square of (6) to (5) is the power rati we wish to maximize:

$$\rho = \frac{2\left[\int_{-\infty}^{\infty} S(j2\pi f)G(j2\pi f)e^{j2\pi f\Delta}\,df\right]^2}{N_0\int_{-\infty}^{\infty} |\,G(j2\pi f)\,|^2\,df}. \tag{7}$$

Recognizing that the integral in the numerator is real [it is $y_s(\Delta)$], and identifying $G(j2\pi f)$ with $f(x)$ and $S(j2\pi f)e^{j2\pi f\Delta}$ with $g(x)$ in the Schwarz inequality,

$$\left|\int f(x)g(x)\,dx\right|^2 \le \int |\,f(x)\,|^2\,dx \int |\,g(x)\,|^2\,dx, \tag{8}$$

one obtains from (7)

$$\rho \le \frac{2}{N_0}\int_{-\infty}^{\infty} |\,S(j2\pi f)\,|^2\,df. \tag{9}$$

Since $|\,S(j2\pi f)\,|^2$ is the energy density spectrum of $s(t)$ the integral in (9) is the total energy, E, in $s(t)$. Then

$$\rho \le \frac{2E}{N_0}. \tag{10}$$

It is clear on inspection that the equality in (8), and hence in (9) and (10), holds when $f(x) = kg^*(x)$, *i.e.*, when

$$G(j2\pi f) = kS^*(j2\pi f)e^{-j2\pi f\Delta}. \tag{11}$$

Thus, when the filter is matched to $s(t)$, a maximum value of ρ is obtained. It is further easily shown that the equality in (8) holds *only* when $f(x) = kg^*(x)$, so the matched filter of (11) represents the only type of linear filter which maximizes ρ.

Notice that we have assumed nothing about the statistics of the noise except that it is stationary and white with power density $N_0/2$. If it is not white, but has some arbitrary power density spectrum $|\,N(j2\pi f)\,|^2$, a derivation similar to that given above [9], [15] leads to the solution

$$G(j2\pi f) = \frac{kS^*(j2\pi f)e^{-j2\pi f\Delta}}{|\,N(j2\pi f)\,|^2}. \tag{12}$$

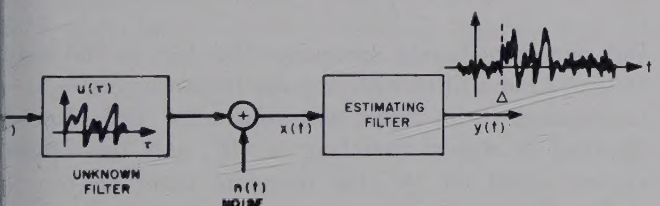
[2] This is a *density* spectrum: that is, if $s(t)$ is, *e.g.*, a voltage waveform, $S(j2\pi f)$ is a voltage density, and its integral from f_1 to f_2 (plus that from $-f_2$ to $-f_1$) is the part of the voltage in $s(t)$ originating in the band of frequencies from f_1 to f_2.

[3] This is the "double-ended" density, covering positive and negative frequencies. The "single-ended" physical power density (positive frequencies only) is thus N_0.

ne can convince himself of this intuitively in the folving manner. If the input, $x(t)$, of Fig. 2 is passed rough a filter with transfer function $1/N(j2\pi f)$, the ise component at its output will be white; however, e signal component will be distorted, now having the ectrum $S(j2\pi f)/N(j2\pi f)$. On the basis of our previous scussion of signals in white noise, it seems reasonable, en, to follow the noise-whitening filter with a filter atched to the distorted signal spectrum, *i.e.*, with the ter $kS^*(j2\pi f)e^{-j2\pi f\Delta}/N^*(j2\pi f)$. The cascade of the noiseitening filter and this matched filter is indeed the lution (12).[4]

So far we have considered only a detection problem: the signal present or not? Suppose, however, we know at the signal is present, but has an unknown delay, which we wish to measure (*e.g.*, radar ranging). Then e first of the output waveforms in Fig. 2 obtains, but e peak in it is delayed by the unknown delay. In order measure this delay accurately, we should not only e the output waveform, as before, to be large at $t =$ $+ t_0$ but also to be very small elsewhere.

More generally, we may frame the problem in the anner depicted in Fig. 3. A sounding signal of known rm, $s(t)$, is transmitted into an unknown filter, the pulse response of which is $u(\tau)$. [In Fig. 2, this filter merely a pair of wires with impulse response $\delta(t)$, the rac delta function.] At the output of the unknown er, stationary noise is added to the signal, the sum ing denoted by $x(t)$. We desire to operate on $x(t)$ with inear filter, whose output, $y(t)$, is to be as faithful as ssible an estimate of the unknown impulse response, rhaps with some delay Δ.



. 3—Pertaining to mean-square estimation of an unknown impulse response.

The unknown filter may be some linear or quasi-linear nsmission medium, such as the ionosphere, the characistics of which we wish to measure. Again, it may resent a complex of radar targets, in which case $u(\tau)$ y consist of a sequence of delta functions with unknown ays (ranges) and strengths.

reasonable mean-square criterion for faithfulness of roduction is that the average of the squared difference ween $y(t)$ and $u(t - \Delta)$, integrated over the pertinent ge of t, be as small as possible; the average must be en over both the ensemble of possible impulse responses of the unknown filter, and the ensemble of possible noises. Using such a criterion [51], one arrives at an optimum estimating filter which is in general relatively complicated. However, when the signal-to-noise ratio is small, and it is assumed that nothing whatever is known about $u(\tau)$ except possibly its maximum duration, the optimum filter turns out to be matched to $s(t)$ if the noise is white, and has the form of (12) for nonwhite noise. Further, in the important case when the sounding signal, $s(t)$, is also optimized to minimize the error in $y(t)$, the optimum estimating filter is matched to $s(t)$ for all signal-to-noise ratios and for all degrees of *a priori* knowledge about $u(\tau)$, provided only that the noise is white.[5]

B. Probabilistic Criteria

In the preceding discussion, we have confined ourselves to mean-square criteria—maximization of a signal-to-noise power ratio or minimization of a mean-square difference. The use of such simple criteria has the advantage of not requiring us to know more than a second-order statistic of the noise—the power-density spectrum. But, although mean-square criteria often have strong intuitive justifications, we should prefer to use criteria directly related to performance ratings of the systems in which we are interested, such as radar and communication systems. Such performance ratings are usually probabilistic in nature: one speaks of the probabilities of detection, of false alarm, of error, etc., and it is these which we wish to optimize. This brings us into the realm of classical statistical hypothesis-testing and estimation theories.

Let us first examine perhaps the simplest hypothesistesting problem, the one posed at the start of the section on mean-square criteria: the observed signal, $x(t)$, is either due solely to noise, or to both an exactly known signal and noise.[6] Such a situation could occur, for example, in an on-off communication system, or in a radar detection system. Adopting the standard parlance of hypothesistesting theory, we denote the former hypothesis, noise only, by H_0, and the alternative hypothesis by H_1. We wish to devise a test for deciding in favor of H_0 or H_1.

There are two types of errors with which we are concerned: a Type I error, of deciding in favor of H_1 when H_0 is true, and a Type II error, of deciding in favor of H_0 when H_1 is true. The probabilities of making such errors are denoted by α and β, respectively. For a choice of criterion, we may perhaps decide to minimize the average of α and β (*i.e.*, the over-all probability of error); this would require a knowledge of the *a priori* probabilities of H_0 and H_1, which is generally available in a communication system. On the other hand, perhaps one type of error is more costly than the other, and we may then wish to minimize an average cost [29]. When, as is often the case in radar detection, neither the *a priori*

The weak link in this heuristic argument is, of course, that not obvious that an optimization performed on the output of noise-whitening filter is equivalent to one performed on its t, the observed waveform; it can be shown, however, that is so.

[5] Another approach to this problem of impulse-response estimation appears in [25].

[6] Here, however, we shall not initially restrict ourselves as before solely to additive combinations of signal and noise.

probabilities nor the costs are known, or even definable, one often alternatively accepts the criterion of minimizing β (in radar: maximization of the probability of detection) for a given, predetermined value of α (false-alarm probability)—the Neyman-Pearson criterion [7].

What is important for our present considerations is that all these criteria lead to the same generic form of test. If one lets $p_0(x)$ be the probability (density) that *if* H_0 is true, the observed waveform, $x(t)$, could have arisen; and $p_1(x)$ be the probability (density) that *if* H_1 is true, $x(t)$ could have arisen; then the test has the form [7], [29], [33]:

$$\left.\begin{array}{lll}\text{accept} & H_1 & \text{if} \quad \dfrac{p_1(x)}{p_0(x)} > \lambda \\ \text{accept} & H_0 & \text{if} \quad \dfrac{p_1(x)}{p_0(x)} \leq \lambda\end{array}\right\}. \qquad (13)$$

Here λ is a constant dependent on *a priori* probabilities and costs, if these are known, or on the predetermined value of α in the Neyman-Pearson test; most importantly, it is *not* dependent on the observation $x(t)$. The test (13) asks us to examine the possible causes of what we have observed, and to determine whether or not the observation is λ times more likely to have occurred if H_1 is true than if H_0 is true; if it is, we accept H_1 as true, and if not, we accept H_0. If $\lambda = 1$, for example, we choose the cause which is the more likely to have given rise to $x(t)$. A value of λ not equal to unity reflects a bias on the part of the observer in favor of choosing one hypothesis or the other.

Let us assume now that the noise, $n(t)$, is additive, gaussian and white with spectral density $N_0/2$,[3] and further that the signal, if present, has the known form $s(t - t_0)$, $t_0 \leq t \leq t_0 + T$, where the delay, t_0, and the signal duration, T, are assumed known. Then, on observing $x(t)$ in some observation interval, I, which includes the interval $t_0 \leq t \leq t_0 + T$, the two hypotheses concerning its origin are:

$$\left.\begin{array}{l} H_0: \ x(t) = n(t), \quad t \ \text{in} \ I \\ H_1: \ x(t) = s(t - t_0) + n(t), \quad t \ \text{in} \ I\end{array}\right\}. \qquad (14)$$

Now, it can be shown [57] that the probability density of a sample, $n(t)$, of white, gaussian noise lasting from a to b may be expressed as[7]

$$p(n) = k \exp\left[-\frac{1}{N_0}\int_a^b n^2(t)\, dt\right], \qquad (15)$$

where $N_0/2$ is the double-ended spectral density of the noise, and k is a constant not dependent on $n(t)$. Hence the likelihood that, if H_0 is true, the observation $x(t)$ could arise is simply the probability (density) that the noise waveform can assume the form of $x(t)$, *i.e.*,

$$p_0(x) = k \exp\left[-\frac{1}{N_0}\int_I x^2(t)\, dt\right], \qquad (16)$$

[7] The space on which this probability density exists must be carefully defined, but the details of this do not concern us here.

the region of integration being, as indicated, the observ tion interval, I. Similarly, the likelihood that, if H_1 true, $x(t)$ could arise is the probability density that t noise can assume the form $n(t) = x(t) - s(t - t_0)$, *i.*

$$\begin{aligned} p_1(x) &= k \exp\left[-\frac{1}{N_0}\int_I [x(t) - s(t - t_0)]^2\, dt\right] \\ &= k \exp\left[-\frac{1}{N_0}\int_I x^2(t)\, dt \right. \\ &\qquad \left. + \frac{2}{N_0}\int_I s(t - t_0)x(t)\, dt - \frac{E}{N_0}\right], \qquad (1 \end{aligned}$$

where we have denoted $\int_I s^2(t - t_0)\, dt$, the energy the signal, by E.

On substituting (16) and (17) in (13) and taking t logarithm of both sides of the inequality, the hypothes testing criterion becomes

$$\left.\begin{array}{lll}\text{accept} & H_1 & \text{if} \quad y(t_0) > \lambda' \\ \text{accept} & H_0 & \text{if} \quad y(t_0) \leq \lambda'\end{array}\right\}, \qquad (1$$

where we have set

$$y(t_0) = \int_I s(t - t_0)x(t)\, dt \qquad (1$$

and

$$\lambda' = \frac{N_0}{2}\log \lambda + 2E. \qquad (2$$

Changing variable in (19) by setting $\tau = t_0 - t$, o obtains

$$y(t_0) = \int_{-T}^{0} s(-\tau)x(t_0 - \tau)\, d\tau. \qquad (2$$

But one immediately recognizes this last as the outp at time t_0, of a filter with impulse response $g(\tau) = s(-$ in response to an input waveform $x(t)$. The filter th specified is clearly matched to $s(t)$, and the optimu system called for by (18) therefore takes on the fo of Fig. 4.[8]

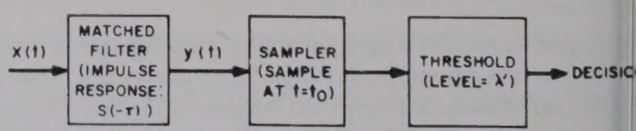


Fig. 4—A simple radar detection system.

[8] Notice that $s(-\tau)$, hence $g(\tau)$, is nonzero only in the inter $-T \leq \tau \leq 0$; hence the limits of integration. In order to rea $g(\tau)$, a delay of $\Delta \geq T$ must be inserted in $g(\tau)$—see (1) and footr 1. This introduces an equal delay at the filter output, which m then be sampled at time $t_0 + \Delta$, rather than at t_0.

Notice also that $y(t_0)$ could be obtained by literally follow the edicts of (19). That is, one could multiply the incoming wa form, $x(t)$, by a stored replica of the signal waveform, $s(t)$, dela by t_0; the product, integrated over the observation interval $y(t_0)$, and is to be compared with the threshold, λ'. Such a dete is called a correlation detector. We shall in this paper, howe refer solely to the matched-filter versions of our solutions, w the understanding that these may be obtained by correla techniques if desired.

The similarity between the solution represented in ig. 4 and the one we obtained in connection with Fig. 2 apparent: the matched filter in Fig. 4 in fact maximizes ne signal-to-noise ratio of $y(t)$ at $t = t_0$, the sampling nstant. That a mean-square criterion and a probabilistic riterion should lead to the same result in the case of dditive, gaussian noise is no coincidence; there is an ntimate connection between the two types of criteria in his case.

So far, we have considered that the signal component $x(t)$, if it is present, is known exactly; in particular, e have assumed that the delay t_0 is known. Suppose ow that the envelope delay of the signal is known, but ot the carrier phase, θ [50]. Assuming that a probability nsity distribution is given for θ, (13) reduces to

$$\left.\begin{array}{ll}\text{accept } H_1 \quad \text{if} & \dfrac{\displaystyle\int_0^{2\pi} p_1(x/\theta)p(\theta)\,d\theta}{p_0(x)} > \lambda'' \\ \text{accept } H_0 \quad \text{otherwise} & \end{array}\right\}. \tag{22}$$

ere $p_1(x/\theta)$ is the conditional probability, given θ, that H_1 is true $x(t)$ will arise; $p(\theta)$ is the probability density θ. If θ is completely random, so $p(\theta)$ is flat, then carrying rough the computations for the case of white, gaussian ise leads to the intuitively expected result that an velope detector should be inserted in Fig. 4 between e matched filter and the sampler [38], [57]. Note, hower, that the matched filtering of $x(t)$ is still the core of e test.

If the envelope delay is also unknown, and no probility distribution is known for it *a priori* other than at it must lie in a given interval, Ω, a good test is [7]:

$$\left.\begin{array}{ll}\text{cept} \quad H_1 \quad \text{if} & \dfrac{\displaystyle\max_{\Omega}\int_0^{2\pi} p_1(x/\theta)p(\theta)\,d\theta}{p_0(x)} > \lambda''' \\ \text{cept} \quad H_0 \quad \text{otherwise} & \end{array}\right\}, \tag{23}$$

ere it is implicit that the integral in the numerator pends on a hypothesized value of the envelope delay.[9] r a flat distribution of θ, (23) yields a circuit in which e envelope detector just inserted in Fig. 4 remains, but now followed not by a sampler, but by a wide gate, en during the interval Ω. For all values of t_0 within is interval, this gate passes the envelope of $y(t_0)$ to the reshold; if the threshold is exceeded at any instant, en the maximum value of the gate output will also rely exceed it, and, by (23), H_1 must then be accepted.

[9] Here, for lack of knowing the true envelope delay, we have entially used test (22), in which we have assumed that the enope delay has that value which maximizes the integral in the nerator. This procedure is part and parcel of the estimation blem, which we have already considered briefly in the discussion mean-square criteria, and which we shall later bring into the sent discussion on probabilistic criteria.

As before, a matched filtering operation is the basic part of the test.[10]

We have been discussing, mostly in the radar context, the simple binary detection problem, "signal plus noise or noise only?" Clearly, this is a special case of the general binary detection problem, more germane to communications, "signal 1 plus noise or signal 2 plus noise?", where we have taken one of the signals to be identically zero. Let us now address ourselves to the even more general digital communications problem, "which of M possible signals was transmitted?"

Let us first consider the case in which the forms of the signals, *as they appear at the receiver*, are known exactly. In this situation, a good hypothesis test, of which (13) is a special case, is of the form:

$$\left.\begin{array}{l}\text{accept the hypothesis,} \quad H_m, \quad \text{for which} \\ \mu_m p_m(x) \ (m = 1, \cdots, M) \quad \text{is greatest}\end{array}\right\}. \tag{24}$$

Here $p_m(x)$ is the probability that if the mth signal was sent, $x(t)$ will be received. The μ_m's are constants, independent of $x(t)$, which are determined solely by the criterion of the test. If the criterion is, for example, minimization of the over-all probability of error, μ_m is the *a priori* probability of transmittal of the mth signal [56], [57]. The μ_m's may possibly also be related to the costs of making various types of errors. If neither *a priori* probabilities nor costs are given, the μ_m's are generally all equated to unity, and the test is called a maximum-likelihood test [7].

If the noise is additive, gaussian, and white with spectral density $N_0/2$, then following the reasoning which led to (17), we have

$$p_m(x) = k \exp\left[-\frac{1}{N_0}\int_I x^2(t)\,dt + \frac{2}{N_0}\,y_m(0) - \frac{E_m}{N_0}\right], \tag{25}$$

where, letting $s_m(t)$ be the mth signal,

$$y_m(t) = \int_{-T}^{0} s_m(-\tau)x(t-\tau)\,d\tau. \tag{26}$$

We have arbitrarily assumed that the signals are received with no delay (this amounts to choosing a time origin), and that they all have the same duration, T. E_m is the energy in the mth signal.

Taking logarithms for convenience, test (24) reduces to:

$$\left.\begin{array}{l}\text{accept the hypothesis,} \quad H_m, \quad \text{for which} \\ y_m(0) + \left(\dfrac{N_0}{2}\log\mu_m - 2E_m\right) \text{ is greatest}\end{array}\right\}. \tag{27}$$

[10] Note that if a correlation detector (see footnote 8) is to be used here, it must compute the envelope of $y(t_0)$ separately for each value of t_0 in Ω. That is, (19) *and its quadrature component* must be obtained, squared and summed *for each* t_0 by a separate multiplication and integration process; t_0 is here a parameter (parametric time). In general, the advantages of using a matched filter to obtain $y(t_0)$, for all t_0 in Ω, as a function of real time are obviously great. In a few circumstances, however—such as range-gated pulsed radar—the correlation technique may be more easily implemented, to a good approximation.

$y_m(0)$ is, as we have seen, the output at $t = 0$ of a filter matched to $s_m(t)$, in response to the input $x(t)$. Therefore, the optimum receiver has the configuration of Fig. 5, where the biases B_m are the quantities $(N_0/2) \log \mu_m - 2E_m$. The output of the decision circuit is the index m which corresponds to the greatest of the M inputs.

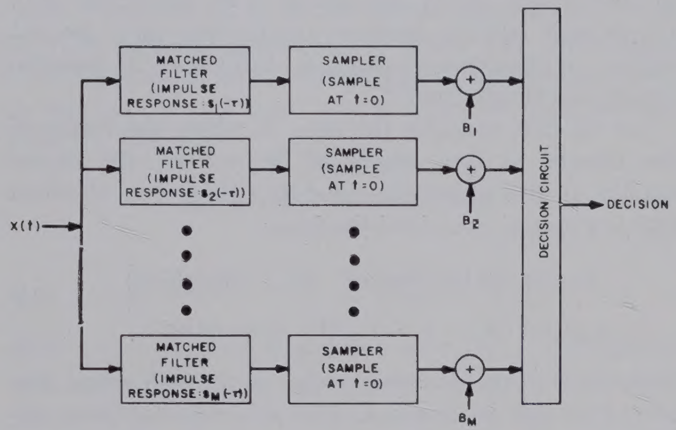


Fig. 5—A simple M-ary communications receiver.

If the forms of the signals $s_m(t)$, as they appear at the receiver, are not known exactly, but are dependent on, say, L random parameters, $\theta_1, \cdots, \theta_L$, whose joint density distribution $q(\theta_1, \cdots, \theta_L)$ is known, then (24) takes the form:

$$\left.\begin{array}{l}\text{accept the hypothesis, } H_m \text{, for which} \\ \mu_m \int \cdots \int q(\theta_1, \cdots, \theta_L) \\ \quad \cdot p_m(x/\theta_1, \cdots, \theta_L)\, d\theta_1 \cdots d\theta_L \quad \text{is greatest}\end{array}\right\}, \qquad (28)$$

where $p_m(x/\theta_1, \cdots, \theta_L)$ is the conditional probability, given $\theta_1, \cdots, \theta_L$, that if H_m is true, the observed received signal, $x(t)$, will arise.

Suppose there is only one unknown parameter, the carrier phase, θ, which is assumed to have a uniform distribution. Then, as in the on-off binary case, it turns out that the only modification required in the configuration of the optimum receiver is that envelope detectors of a particular type[11] must be inserted between the matched filters and samplers in Fig. 5.

Test (28) may also be applied to the case of multipath communications. For example, one may consider the case of a discrete, slowly varying, multipath channel in which the paths are independent, and each is characterized by a random strength, carrier phase-shift, and modulation delay; if P is the number of paths, there are then $3P$ random parameters on which the signals depend. It can be shown [50] that, for a broad class of probability distr butions of path characteristics, an optimum-receiver a rangement similar to Fig. 5 still obtains. Instead of simple sampler following the matched filter, howeve there is in general a more complicated nonlinear devic the form of which depends on the form of the distributio $q(\theta_1, \cdots, \theta_L)$, inserted in (28). If, for instance, it assumed that the modulation delays of the paths a known, but that the path strengths and phase-shif share joint distributions of a fairly general form [50 the device combines nonlinear functions of several sample taken at instants corresponding to the expected pat delays, of both the output of the matched filter and its envelope. Again, if all the paths are assumed *a prio* to have identical strength distributions, their carri phase-shifts assumed uniformly distributed over $(0, 2\pi$ and their modulation delays assumed uniformly distr buted over the interval (t_a, t_b),[12] the form of the nonlinea device which should replace the mth sampler in Fig. is given, to a good approximation, by Fig. 6 [12], [50 The exact form of the nonlinear device, F_m, is determine by the assumed path-strength distribution.

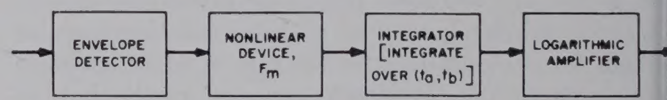


Fig. 6—A nonlinear device for use in Fig. 5 when the path dela are unknown.

The case in which the path modulation delays a known, and the carrier phase-shifts and strengths va randomly *with time* (*i.e.*, are not restricted, as befor to "slow" variations with respect to the signal duration has also been considered for Rayleigh-fading paths [36 Again, at least to an approximation, the results can interpreted in terms of matched filters. The operatio performed on the matched filter outputs now vary wi time, however.

In order to use test (28), it is necessary to have parameter distribution, $q(\theta_1, \cdots, \theta_L)$. It may be, howeve that a complete distribution of all the parameters is n available. Then it is often both theoretically and pract cally desirable to return to a test based on the ten that the parameters for which no distribution is giv are exactly known; in place of the true values of the parameters called for by the test, however, estimates a used. We have already come across such a technique connection with (23); in that case, an *a priori* distributi of one parameter, phase, was used, but another paramete modulation delay, was estimated to be that value whi maximized the integral in the numerator.[9]

[11] Unlike the on-off binary case, in which any monotone-increasing transfer characteristic can be accommodated in the envelope detector by an adjustment of the threshold, in this case, except when all the biases are equal, the transfer characteristic must be of the log I_0 type [7], [38], [50]. In the region of small signal-to-noise ratios, however, such a characteristic is well approximated by a square-law detector.

[12] These distributions of phase shift and modulation delay m not, of course, be totally due to randomness in the channel, but m partially reflect a distribution of error in phase-base and time-ba synchronization of transmitter and receiver; the original assumpti of knowledge of the *exact* forms of the signals as they appear the receiver implied perfect synchronization.

Let us, as a final topic in our discussion of probabilistic criteria, then, consider this problem of estimation of parameters. Clearly, estimation may be an end in itself, as in the case of radar ranging briefly considered in the section of mean-square criteria; on the other hand, it may be to the end discussed in the preceding paragraph.

The step from hypothesis testing to estimation may in a sense be considered identical to that from discrete to continuous variables. In the hypothesis test (24), for example, one inquires, "To which of M discrete causes (signals) may the observation $x(t)$ be ascribed?" If one is alternatively faced with a continuum of causes, such as a set of signals, identical except for a delay which may have any value within an interval, the equivalent test would clearly be of the form:

$$\left.\begin{array}{l}\text{accept the hypothesis,} \quad H_m, \quad \text{for which} \\ \mu(m)p_m(x) \ (m_1 \leq m \leq m_2) \quad \text{is greatest}\end{array}\right\}, \tag{29}$$

where the notation is the same as that used in (24), except that m is now a continuous parameter. Only one parameter has been explicitly shown in (29), but clearly the technique may be extended to many parameters.

To consider but one application of (29), let us again investigate the radar-ranging problem, in which the observation is known to be given by

$$x(t) = s(t - t_0) + n(t) \quad \begin{array}{c} a \leq t_0 \leq b \\ t \ \text{in} \ I \end{array}, \tag{30}$$

where $s(t)$ is of known form, nonzero in the interval $0 \leq t \leq T$. Again let us imagine $n(t)$ to be stationary, gaussian and white with spectral density $N_0/2$. Then a little thought will convince one that the probability that $x(t)$ will be observed, if the delay has some specific value t_0 in the interval (a, b), is of the form of the right-hand side of (17). Placing this in (29), identifying m with t_0, and letting $\mu(m)$ be independent of m (maximum-likelihood estimation) [7], it becomes clear that we seek the value of t_0 for which the integrals (19) and (21) are maximum. Remembering our previous interpretation of (21), it follows that a system which estimates delay in this case is one which passes the observed signal, $x(t)$, into a filter matched to the signal $s(t)$, and estimates, as the delay of $s(t)$, the delay (with respect to a reference time) of the maximum of the filter's output. Thus, in optimum radar systems, at least in the case of additive, gaussian noise, a matched filter plays a central part in both the detection and ranging operations. This is a dual rôle we have previously had occasion to note heuristically.[9]

The application of the technique of estimation of unknown signal parameters for use in a hypothesis test has been considered for ideal multipath communication systems [21], [36], [37], [41], [48], [50]. In fact, even if such estimation techniques are not postulated explicitly to start with, but some *a priori* distribution of the unknown parameters is assumed, it is on occasion found that operations which may be interpreted as implicit *a posteriori* estimation procedures appear in the hypothesis-testing receiver [21], [36]. Again, matched filters, or operations very close to matched filtering operations, arise prominently in the solutions.

It is perhaps not necessary to emphasize that the point of the foregoing discussion of where matched filters arise is not to paint a detailed picture of optimum detection and estimation devices; this has not been done, nor could it, in the limited space available. Rather, the point is to indicate that matched filters do appear as the core of a large number of such devices, which differ largely in the type of nonlinear operations applied to the matched-filter output. Having made this point, it behooves us to study the properties of matched filters and to discuss methods of their synthesis, to which topics the remainder of this paper will be devoted.

It seems only proper, however, to end this section with a caveat. The devices based on probabilistic criteria in which we have encountered matched filters were derived on the assumption that at least the additive part of the channel disturbance is stationary, white and gaussian. The stationarity and whiteness requirements are not too essential; the elimination of the former generally leads to time-varying filters, and the elimination of the latter, to the use of the noise-whitening pre-filters we have previously discussed [11], [30]. The requirement of gaussianness is not as easy to dispense with. We have seen that, within the bounds of linear filters, a matched filter maximizes the signal-to-noise ratio for any white, additive noise, and this gives us some confidence in their efficacy in nongaussian situations. But due care should nonetheless be exercised in implying from this their optimality from the point of view of a probabilistic criterion.

IV. Properties of Matched Filters

In order to gain an intuitive grasp of how a matched filter operates, let us consider the simple system of Fig. 7.

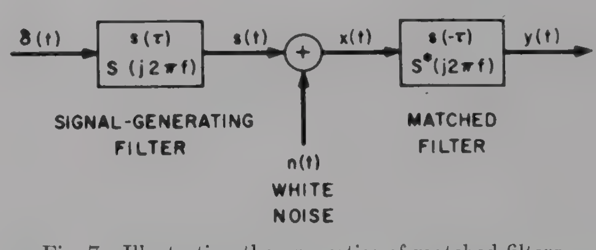


Fig. 7—Illustrating the properties of matched filters.

A signal, $s(t)$, say of duration T, may be imagined to be generated by exciting a filter, whose impulse response is $s(\tau)$, with a unit impulse at time $t = 0$. To this signal is added a white noise waveform, $n(t)$, the power density of which is $N_0/2$. The sum signal, $x(t)$, is then passed into a filter, matched to $s(t)$, whose output is denoted by $y(t)$.

This output signal may be resolved into two components,

$$y(t) = y_s(t) + y_n(t), \tag{31}$$

the first of which is due to $s(t)$ alone, the second to $n(t)$ alone. It is these two components which we wish to study. For simplicity of illustration, let us take all spectra to be centered around zero frequency; no generality is lost in our results by doing this.

The response to an input $s(t)$ of a linear filter with impulse response $h(\tau)$ is

$$\int_{-\infty}^{\infty} h(\tau)s(t - \tau)\, d\tau.$$

If $h(\tau) = s(-\tau)$, as in Fig. 7, then

$$y_s(t) = \int_{-\infty}^{\infty} s(-\tau)s(t - \tau)\, d\tau. \tag{32}$$

This is clearly symmetric in t, since

$$\begin{aligned} y_s(-t) &= \int_{-\infty}^{\infty} s(-\tau)s(-t - \tau)\, d\tau \\ &= \int_{-\infty}^{\infty} s(t - \tau')s(-\tau')\, d\tau' = y_s(t), \end{aligned} \tag{33}$$

where the second equality is obtained through the substitution $\tau' = t + \tau$. For the low-pass case we are considering, $y_s(t)$ may look something like the pulse in Fig. 8. The height of this pulse at the origin is, from (32),

$$y(0) = \int_{-\infty}^{\infty} s^2(\tau)\, d\tau = E, \tag{34}$$

where E is the signal energy. By applying the Schwarz inequality, (8), to (32), it becomes apparent that $| y_s(t) |$ cannot exceed $y(0) = E$ for any t.

The spectrum of $y_s(t)$, that is, its Fourier transform, is $| S(j2\pi f) |^2$. This is easily seen from the fact that $y_s(t)$ may be looked on, in Fig. 7, as the impulse response of the cascade of the signal-generating filter and the matched filter. But the over-all transfer function of the cascade, which is therefore the spectrum of $y_s(t)$, is just $| S(j2\pi f) |^2$. The spectrum corresponding to the $y_s(t)$ in Fig. 8 will look something like Fig. 9.

In Figs. 8 and 9 we have denoted the "widths" of $y_s(t)$ and $| S(j2\pi f) |^2$ by α and β, respectively. It has been shown [13] that for suitable definitions of these widths, the inequality,

$$\alpha\beta \geq \text{a constant of the order of unity}, \tag{35}$$

holds. The exact value of the constant depends on the definition of "width" and need not concern us here. The important thing is that the "width" of the signal component at the matched-filter output in Fig. 7 cannot be less than the order of the reciprocal of the signal bandwidth.

In view of the above results, one might question the optimal character of a matched filter. In the preceding section we saw, for example, that in the radar case with gaussian noise we were to compare $y(t)$ with a threshold to determine whether a signal is present or absent (see Fig. 4). Now, if a signal component *is* present in $x(t)$, we seemingly should require, for the purpose of assuring

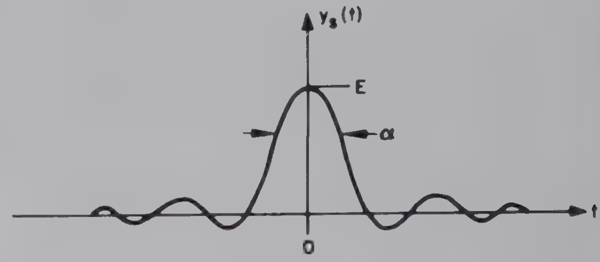


Fig. 8—The signal-component output of (32).

Fig. 9—The spectrum corresponding to $y_s(t)$ in Fig. 8.

that the threshold is exceeded, that the signal component of $y(t)$ be made as large as possible at some instant. Again, if $s(t)$ has been delayed by some unknown amount t_0, so that the signal component of $y(t)$ is also delayed, we saw that we could estimate t_0 by finding the location of the maximum of $y(t)$; in order to find this maximum accurately, one would imagine we would require that the signal component of $y(t)$ be a high, narrow pulse centered on t_0. In short, it appears that for efficient radar detection and ranging we really require an output signal component which looks very much like an *impulse*, rather than like the finite-height, nonzero-width, matched-filter output pulse of Fig. 8. Furthermore, we could actually obtain this impulse by using an inverse filter, with transfer function $1/S(j2\pi f)$, instead of the matched filter in Fig. 7. (To see this, note from Fig. 7 that the output signal component would then be the impulse response of the cascade of the signal-generating filter and the inverse filter, and since the product of the transfer functions of these two filters is identically unity, this impulse response would itself be an impulse.) We are thus led to ask, "Why not use an inverse filter rather than a matched filter?"

The answer is fairly obvious once one considers the effect of the additive noise, which we have so far explicitly neglected in this argument. Any physical signal must have a spectrum, $S(j2\pi f)$, which approaches zero for large values of f, as in Fig. 9. The gain of the inverse filter will therefore become indefinitely large as $f \to \infty$. Since the input noise spectrum is assumed to extend over all frequencies, the power in the output noise component of the inverse filter will be infinite. Indeed, the output noise component will override the output signal-component impulse, as may easily be seen by a comparison of their spectra: the former spectrum increases without limit as $f \to \infty$, while the latter is a constant for all f.

Thus, one must settle for a signal-component output pulse which is somewhat less sharp than the impulse delivered by an inverse filter. One might look at this as a compromise, a modification of the inverse filter which, while keeping the output signal component as close to an impulse as possible, efficiently suppresses the noise "outside" the signal band. In order to understand the nature of this modification, let us write the signal spectrum in the form

$$S(j2\pi f) = | S(j2\pi f) | e^{-j\psi(f)}, \tag{36}$$

where $\psi(f)$ is the phase spectrum of $s(t)$. Then the inverse filter has the form

$$\frac{1}{S(j2\pi f)} = \frac{1}{| S(j2\pi f) |} e^{j\psi(f)}. \tag{37}$$

Now, since the input noise at any frequency has random phase anyway, we clearly can achieve nothing in the way of noise suppression by modifying the phase characteristic of the inverse filter—we would only distort the output signal component. On the other hand, it seems reasonable to adopt as the amplitude characteristic of the filter, not the inverse characteristic $1/| S(j2\pi f) |$ of (37), but a characteristic which is small at frequencies where the signal is small compared to the noise, and large at the frequencies where the signal is large compared to the noise. In particular, a reasonable choice seems to be $| S(j2\pi f) |$.[13] Then the compromise filter is of the form

$$H(j2\pi f) = | S(j2\pi f) | e^{j\psi(f)} = S^*(j2\pi f), \tag{38}$$

a matched filter. [The last equality in (38) follows from (36) and the fact that, for a physical signal, $| S(j2\pi f) |$ is even in f.]

The compromise solution of (38), of course, as we have already seen, maximizes the height of the signal-component output pulse *with respect to the rms noise output*. This maximum output signal-to-noise ratio turns out to be perhaps the most important parameter in the calculation of the performance of systems using matched filters [20], [28], [33]–[35], [39], [52], [53], [57]; it is given by the right-hand side of (10):

$$\rho_0 = \frac{2E}{N_0}. \tag{39}$$

Note that ρ_0 depends on the signal *only through its energy*, E; such features of the signal as peak power, time duration, waveshape, and bandwidth do not directly enter the expression. In fact, insofar as one is considering *only* the ability of a radar detection system or an on-off communication system to combat white gaussian noise, it follows from this observation that all signals which have the same energy are equally effective.[14]

[13] This is indeed the solution Brennan obtains in deriving weights for optimal linear diversity combination [2]; here we have essentially a case of coherent frequency diversity.

[14] As we shall see, for more complicated communication systems in which more than one signal waveform may be transmitted, parameters governing the "similarity" of the signals enter the performance calculations [20], [52].

One may relate the output signal-to-noise ratio, ρ_0, to that at the input of the filter. Let the noise bandwidth of the matched filter—*i.e.*, the bandwidth of a rectangular-band filter, with the same maximum gain, which would have the same output noise power as the matched filter—be denoted by B_N. Then, for simplicity, one may think of the amount of input noise power within the matched filter "band" as being given by $N_{in} = B_N N_0$. Further let the average signal power at the filter input be $P_{in} = E/T$, where T is the effective duration of the signal. Then, letting $\rho_i = P_{in}/N_{in}$, (39) becomes

$$\rho_0 = 2B_N T \rho_i . \tag{40}$$

In this formulation of ρ_0, it is apparent that the matched filter effects a gain in signal-to-noise *power* ratio of $2B_N T$.

This last result seems at first to contradict our previous observation that, in the face of white noise, the signal bandwidth and time duration do not directly influence ρ_0. There is no contradiction, of course. For a given total signal energy, the larger T is, the smaller the input average signal power P_{in} is, and hence the smaller ρ_i is. Any increase in the ratio of ρ_0 to ρ_i caused by "spreading out" a fixed-energy signal is thus exactly offset by a decrease in ρ_i. Similarly, any increase in the ratio of ρ_0 to ρ_i occasioned by increasing the signal bandwidth is also offset by a decrease in ρ_i; for, the larger the signal bandwidth, the larger B_N, and hence the larger N_{in}—that is, the greater the amount of input noise taken in through the increased matched-filter bandwidth.

However, in relation to the foregoing argument let us suppose that one is combating *band-limited* white noise of *fixed* total power, rather than true white noise, which has infinite total power. That is, suppose the interference is such that its total power N_{in} is always caused, malevolently, to be spread out evenly over the signal "bandwidth" B_N, *whatever* this bandwidth is. Then, in (40), ρ_i is no longer dependent on B_N, and it is clearly advantageous to make the signal and matched-filter bandwidths as large as possible; the larger the bandwidth, the more thinly the total interfering power must be spread, *i.e.*, the smaller the value of N_0 in (39) becomes. In this situation, of all signals with the same energy, the one with the largest bandwidth is the most useful.

An interesting way of looking at the functions of the pair of filters, $S(j2\pi f)$ and $S^*(j2\pi f)$, in Fig. 7 is from the point of view of "coding" and "decoding." The impulse at the input to the signal-generating filter has components at all frequencies, but their amplitudes and phases are such that they add constructively at $t = 0$, and cancel each other out elsewhere. The transfer function $S(j2\pi f)$ "codes" the amplitudes and phases of these frequency components so that their sum becomes some arbitrary waveform lasting, say, from $t = 0$ to $t = T$, such as is shown in Fig. 10. Now, what we should like to do at the receiver is to "decode" the signal, *i.e.*, restore all the amplitudes and phases to their original values. We have seen that we cannot do this, since it would entail an inverse filter; we compromise by restoring the phases,

so that all frequency components at the filter output have zero phase at the same time ($t = 0$ in Fig. 8) and add constructively to give a large pulse. This pulse has a nonzero width, of not less than the order of the reciprocal of the signal bandwidth, because we are not able to restore the amplitudes of the components properly.

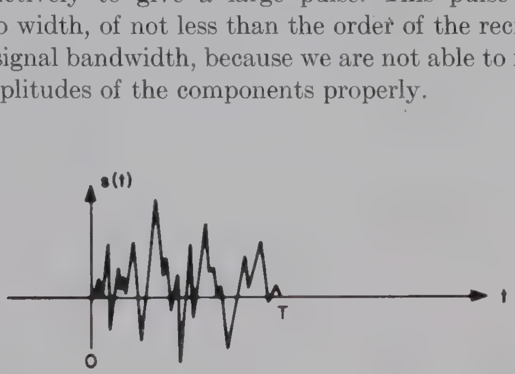


Fig. 10—A signal waveform.

In "coding," then, we have spread the signal energy out over a duration T; in "decoding," we are able to collapse this energy into a pulse of the order of βT times narrower, where β is some appropriate measure of the signal bandwidth. The "squashing" of the energy into a shorter pulse leads to an enhancement of signal-to-noise power ratio by a factor of the order of $B_N T$, already noted in connection with (40).

We thus see that the time-bandwidth product of teh signal or its matched filter, which we shall henceforth denote by TW, is a very important parameter in the description of the filter. It also turns out to be an index of the filter's complexity, as can be seen by the following argument. It is well known [43], that roughly $2TW$ independent numbers are sufficient (although not always necessary) to describe a signal which has an effective time duration T and an effective bandwidth W. It follows that the complete specification of a filter which is matched to such a signal requires, at least in theory, no more than $2TW$ numbers. Therefore, in synthesizing the filter there theoretically need be no more than $2TW$ elements or parameters specified, whence the use of the TW product as a measure of "complexity." (See footnote 20, however.)

Hopefully, this section has provided an intuitive insight into the nature of matched filters. It is now time to investigate the problems encountered in their synthesis.

V. Matched-Filter Synthesis and Signal Specification

In considering the synthesis of matched filters, one must take account of the edicts of two sets of constraints: those of physical realizability, and those of what might be called practical realizability. The first limit what one *could* do, at least in theory; the second are more realistic, for they recognize that what is theoretically possible is not always attainable in practice—they define the limits of what one *can* do.

The constraints of physical realizability are relatively easily given, and may be found in any good book on network synthesis [19]. Perhaps the most important for us, at least for electrical filters, is that expressed in footnote 1, that the impulse response must be zero for negative values of its argument.[15] If the signal to which the filter is to be matched "stops"—*i.e.*, falls to zero and remains there thereafter—at some finite time t_2 (see Fig. 1), then we have seen that by introducing a finite but perhaps large delay, $\Delta \geq t_2$, in the impulse response, we can render the impulse response physically realizable. We must of course then wait until $t = \Delta \geq t_2$ for the peak of the output signal pulse to occur; put another way, we cannot expect an output containing the full information about the signal at least until the signal has been fully received. Suppose, however, that t_2 is infinite, or it is finite but we cannot afford to wait until the signal is fully received before we extract information about it. Then it may easily be shown, for example, that in order to maximize the output signal-to-noise ratio at some instant $t < t_2$, we should use that part of the optimum impulse response which is realizable, and delete that part which is not [58]. Thus, if the signal of Fig. 1(a) is to be detected at $t = 0$, the *desired* impulse response is proportional to that of Fig. 1(b); the best we can do in rendering this physically realizable is to delete that part of it occurring prior to $\tau = 0$. The output signal-to-noise ratio at the instant of the output signal peak (in this case, $t = 0$) is still of the form of (39), but now E must be interpreted not as the total signal energy, but only as that part of the signal energy having arrived by the time of the output signal peak. Of course, we are no longer dealing with a true matched filter.

The constraints of practical realizability are not so easy to formulate: they are, rather, based on engineering experience and intuition. Let us henceforth assume that we are concerned with a true matched filter, *i.e.*, that the impulse response (1) is physically realizable. Even then, we are aware that the filter may not be practically realizable, because too many elements may be required to build it, or because excessively flawless elements would be needed, or because the filter would be too difficult to align or keep aligned, etc. In this light, the problem of realizing a matched filter changes from "Here is a desirable signal; match a filter to it" to "Here is a class of filters which I can satisfactorily build; which members of the class, if any, correspond to desirable signals?" In the former case we might here have neglected the question of how it was decided that the given signal is "desirable"; it would perhaps have sufficed merely to discuss how to realize the filter. But from the latter point of view the choice of a practical filter becomes inextricably interwoven with criteria governing the desirability of a signal, and one would therefore do well to design both the filter *and* the signal together to do the best over-all job. For this reason it is worthwhile to devote some time to answering the question, "What is a desirable signal?"

[15] Note that such a constraint is not necessary for optical filters [5], and filters—such as may be programmed on a computer—which use parametric rather than real time.

The answer, of course, depends on the application. Let us first consider the case of radar detection and ranging.

From the point of view of detection, we noted in the last section that in the face of white gaussian noise all signals with the same energy are equally effective, while if the interfering noise is band-limited and is constrained to have a given total power, then of all signals of the same energy those with the largest bandwidth are the most effective.

From the point of view of ranging there are at least three properties of the signal which we must consider: *accuracy*, *resolution*, and *ambiguity*. Let us first discuss these for the case of a low-pass signal. In this case, the matched-filter output has the form of the pulse in Fig. 8, but delayed by an unknown amount and immersed in noise. In order accurately to locate the position of the delayed central peak of the output signal pulse, we should like both to make ρ_0 of (39) large and also to make α, the width of this peak, small. This latter requirement implies making the bandwidth large [see (35)], whether the noise is truly white, or is of the band-limited, fixed-power variety. Further, if several targets are present, so that the signal component of the matched filter output consists of several pulses of the form of Fig. 8, delayed by various amounts (see Fig 11), then making α very small will allow closely adjacent targets to be resolved. That is, if two targets have nearly the same delay, as, for example, targets 3 and 4 in Fig. 11, making α small enough will lead to the appearance of two distinct peaks in the matched-filter output, rather than a broad hump.

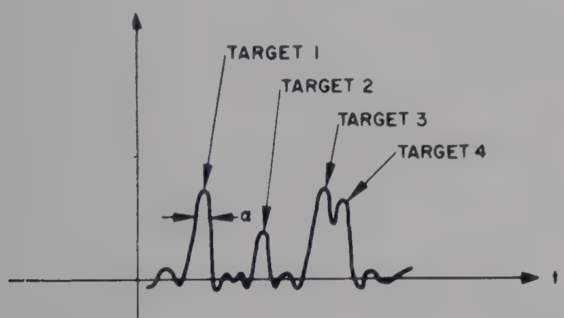


Fig. 11—The signal-component output of a matched filter for a resolvable multitarget or multipath situation.

The requirements of accuracy and resolution thus dictate a large-bandwidth signal. A more stringent requirement is that of lack of ambiguity. To explain this, let us again consider Fig. 8, the output signal component in response to a single target. As shown, there is but one peak, so even if the peak is shifted by an unknown delay there is hope that, despite noise, the amount of the delay can be determined unambiguously. However, suppose now that the waveform of Fig. 8 were to have many peaks of equal height, as would happen if the signal $s(t)$ were periodic [see (32)]. Then even without noise, it might not be totally clear which peak represented the unknown target delay; that is, there would be ambiguity in the target range, a familiar enough phenomenon in, say, periodically pulsed radars.

For the purposes of radar ranging then, what we require in the low-pass case is a large bandwidth signal, $s(t)$, for which, from (32),

$$y_s(t) = \int_{-\infty}^{\infty} s(\tau)s(t+\tau)\,d\tau \tag{41}$$

has a narrow central peak, and is as close to zero as possible everywhere else. The right-hand side of (41) may also be written as the real part of

$$2\int_0^{\infty} |S(j2\pi f)|^2 e^{j2\pi ft}\,df, \tag{42}$$

where $S(j2\pi f)$ is, as usual, the spectrum of $s(t)$. We see, therefore, that we are here concerned with a shaping of the energy density spectrum of the signal.

In the case of a band-pass signal with random phase, all we have said still holds, but now not with respect to $y_s(t)$, but with respect to its envelope; $y_s(t)$ itself now has a fine oscillatory structure at the carrier frequency. This envelope is expressible as the magnitude of (42) [57]. In other words, we now want the *envelope* of the matched-filter output, in the absence of noise, to have a narrow central peak and be as close to zero as possible elsewhere. As before, the minimum width attainable by the central peak is of the order of the reciprocal of the signal bandwidth.

We have thus far in the present paper neglected the possibility of doppler shift in our discussion of radar systems. It is worthwhile to insert a few words on this topic now; we shall limit ourselves, however, to consideration of narrow-band band-pass signals, which are the only ones for which we can meaningfully speak of a doppler "*shift*" of frequency. Suppose, in addition to being delayed, the target return signal may also be shifted in frequency. It turns out then that when additive, white, gaussian noise is present, the ideal receiver should contain a parallel bank of matched filters much like that in the communication receiver of Fig. 5. Each filter is matched to a frequency-shifted version of the transmitted signal, there being a filter for each possible doppler shift. (If there is a continuum of possible doppler shifts, we shall see that the required "continuum" of filters is well approximated by a finite set, in which the frequency shifts are evenly spaced by amounts of the order of $1/T$, the reciprocal of the duration of the transmitted signal.) The (narrow-band) outputs of the bank of matched filters are then all envelope detected and subsequently passed into a device which decides, according to the edicts of hypothesis-testing and estimation theories, whether or not a target is present, and if present, at what delay (range) and doppler shift (velocity).

Now, as in the case of no doppler shift, the detectability of the signal is still governed by (39) and (40). But a modification is required in our previous discussion of the demands of high accuracy, high resolution, and low ambig-

uity. Generalizing this discussion to the case of nonzero doppler shift, it is clear that we require the following: each target represented at the receiver input should excite *only* the filter in the matched-filter bank which corresponds to the target doppler shift (velocity), and, further, should cause a sharp peak to appear in this filter's output envelope *only* at a time corresponding to the delay of the target, and nowhere else. In order to state this requirement mathematically, let us compute the output envelope at time t of a filter matched to a target return with doppler shift ϕ, in response to a target return with zero doppler shift and zero delay. (We lose no generality in assuming these particular target parameters.) We first note that if the (double-sided) spectrum of the transmitted signal is $S(j2\pi f)$, then for the narrow-band band-pass signal we are considering, the spectrum of the signal after undergoing a doppler shift ϕ is approximately $S[j2\pi(f - \phi)]$ *in the positive-frequency region.*[16] A filter matched to this shifted signal therefore has the approximate transfer function $S^*[j2\pi(f - \phi)]$, again in the positive-frequency region. Then the spectrum of the response of this filter to a non-doppler-shifted, nondelayed signal is approximately $S(j2\pi f)\ S^*[j2\pi(f - \phi)]\ (f > 0)$; the response itself, at time t, is the real part of the complex Fourier transform

$$\chi(t, \phi) = 2 \int_0^\infty S(j2\pi f) S^*[j2\pi(f - \phi)] e^{j2\pi f t}\, df. \tag{43}$$

The envelope of the response is just $|\chi(t, \phi)|$, and it is this which we require to be large at $t = 0$ if $\phi = 0$, and small otherwise. That is, we require that $|\chi(t, \phi)|$ have the general shape shown in Fig. 12: a sharp central peak at the origin, and small values elsewhere [23], [44], [57].

Note that (42) is a special case of (43) for $\phi = 0$; hence, the magnitude of (42) corresponds to the intersection of the $|\chi(t, \phi)|$ surface in Fig. 12 with the plane $\phi = 0$. It follows therefore from our discussion of (42) that the central peak in Fig. 12 cannot be narrower than the order of $1/W$ in the t direction. The use of an uncertainty relation of the form of (35) similarly reveals that the "width" of the peak cannot be less than the order of $1/T$ in the ϕ direction, where T is the effective signal duration [57]. (From this is implied a previous statement, that when a continuum of doppler shifts is possible, a good approximation to the ideal receiver involves the use of matched filters spaced apart in frequency by $1/T$; for, a target return which has doppler shift ϕ will cause responses in filters matched to signals with doppler shifts in an interval at least $1/T$ cps wide centered on ϕ.) More generally, one may show that the cross-sectional "area" of the central peak of the surface in Fig. 12—*i.e.*, the size of the (t, ϕ) region over which the peak has appreciable height—cannot be less than the order of $1/TW$ [44]. Thus, in order to attain a very sharp central peak it is

[16] In the negative-frequency region, the shifted spectrum has the approximate form $S[j2\pi(f + \phi)]$.

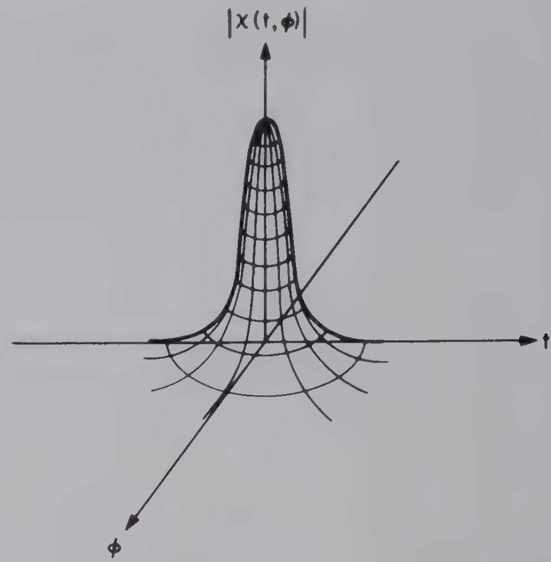


Fig. 12—A desirable $|\chi(t, \phi)|$ function.

necessary—but not sufficient, as we shall see—to make the TW product of the signal very large. Elimination of spurious peaks in $|\chi(t, \phi)|$ away from the (t, ϕ) origin is much more complicated, and has been studied elsewhere [23], [45], [57].

So much for radar signals. The requirements on communication signals are somewhat different, in some ways laxer and in others more stringent. In the simplest case, on-off communication, in which we are concerned with only one signal, the situation is obviously much like radar, except that in general one need not worry about doppler shifts of the transmitted signal. In particular, for gaussian noise, the detectability of the signal, and hence the probability of error, will in this case depend solely on ρ_0 of (39) and (40).

It should be expressly noted, however, that the problems of ranging which we encountered in the radar case are not entirely absent in the communication case. For example, in on-off communication, in order to sample the output of the matched filter at the time its signal component passes through its maximum, we must know when this maximum occurs. In many transmission media, however, the transmitted signal is randomly delayed, thus necessitating a ranging operation, in communication parlance called synchronization. We are not interested in this synchronization time *per se*, as in radar ranging, and are therefore not necessarily interested in eliminating large ambiguous peaks in $y_s(t)$ of (41). If such subsidiary peaks are of the same height as the central one, as will occur if the signal is periodic, we will be just as happy to sample one of them, rather than the central peak. On the other hand, we should not like to synchronize on and sample a peak appreciably smaller than the central peak, for then we would lose in signal-to-noise ratio. Thus, roughly, we require that a spurious peak in $y_s(t)$ be either very large or nonexistent.

We may no longer even require the peaks in $y_s(t)$ to be narrow, for we are not interested in determining the exact location of the peak, but only in sampling the matched-filter output at or near this peak. Clearly, an error in the sampling instant will be less disastrous if the peak is broad than if it is narrow. On the other hand, a narrow peak *is* often desirable in a multipath situation. There are often several independent paths between the transmitter and receiver, which represent independent sources of information about the transmitted signal. It behooves us to keep these sources separate, *i.e.*, to be able to resolve one path from another; this is the same as the radar resolvability requirement, illustrated in Fig. 11. From the frequency-domain viewpoint, requiring the matched-filter output pulse width to be small enough to resolve the various paths is the same as requiring the signal bandwidth to be large enough to avoid nonselective fading of the whole frequency band of the signal.

Such are the considerations which must be given to the choice of a signal for an on-off communication system, or to each signal individually in multisignal systems. However, in systems of the latter type, such as in Fig. 5, one must also consider the relationships *between* signals. In the system of Fig. 5, for example, it is not enough to specify that each signal individually have high energy and excite an output in its associated matched filter which consists, say, of a single narrow pulse at $t = 0$; if this were sufficient, we could choose all the signals to be identical, patently a ridiculous choice. We must also require that the various signals be *distinguishable.* More precisely, if the signal component of $x(t)$ in Fig. 5 is, say, $s_i(t)$, then the signal component at the output of the ith filter should be as large as possible at $t = 0$, *and* at the same instant the signal components of the outputs of all other filters should be "as much different" from the ith filter signal output as possible.

The phrase "as much different" needs defining: one wishes to specify M signals so that the over-all probability of error in reception is minimized. Unfortunately, this problem has not been solved in general for the phase-coherent receiver of Fig. 5. For the special case of binary transmission ($M = 2$), however, it turns out that if the two signals are *a priori* equally probable, one should use equal-energy antipodal signals, *i.e.*, $s_1(t) = -s_2(t)$ [20], [52]. For then, on reception of the ith signal with zero delay, the output signal component of its associated matched filter at $t = 0$ is, from (32),

$$y_s(0) = \int_{-\infty}^{\infty} s_i^2(\tau)\, d\tau = E \qquad (i = 1, 2), \tag{44}$$

where E is the signal energy. The kth filter signal output at $t = 0$ is clearly

$$\int_{-\infty}^{\infty} s_k(-\tau) s_i(-\tau)\, d\tau = -E \qquad (i \neq k), \tag{45}$$

which is easily shown to be "as much different" from (44) as possible.[17]

If one considers the band-pass case in which the carrier phases are unknown, we have seen that the optimum system in Fig. 5 is modified by the insertion of envelope detectors between the matched filters and samplers. In this case a reasonable conjecture, which has been established for the binary case [20], [52], is that the M signals be "envelope-orthogonal." That is, if the ith signal is received, the envelopes of the signal-component outputs of all but the ith filter should be zero at the instant $t = 0$. Now, if the spectrum of the ith signal is denoted by $S_i(j2\pi f)$, then the envelope of the signal-component output of the kth filter is:

$$2 \left| \int_0^{\infty} S_k^*(j2\pi f) S_i(j2\pi f) e^{j2\pi f t}\, df \right| . \tag{46}$$

This, evaluated at $t = 0$, must therefore be zero for all $k \neq i$. There are several ways of assuring that this be so, the most obvious being the use of signals with nonoverlapping bands, so that $S_k^*(j2\pi f)\ S_i(j2\pi f) = 0$.[18] Another way involves the use of signals which are rectangular bursts of sine waves, the sine-wave frequencies of the different signals being spaced apart by integral multiples of $1/T$ cps, where T is the duration of the bursts. A third method is considered elsewhere in this issue [49].

If random multipath propagation is involved and the modulation delays of the paths are known, then we have noted that the samplers in Fig. 5 should in general sample both the matched-filter outputs and their envelopes at several instants, corresponding to the various path delays. We may in this case conjecture, again for the situation of unknown path phase-shifts (*i.e.*, only envelope sampling), that for an optimum set of signals, (46) should vanish for $k \neq i$, but now at values of t corresponding to all path delay *differences*, including zero. For, suppose the ith signal is sent. Then the spectrum

$$\sum_{l=1}^{L} S_i(j2\pi f) e^{-j2\pi f t_l} e^{-j\theta_l} \tag{47}$$

is received, where t_l and θ_l are, respectively, the modulation delay and carrier phase-shift of the lth path [50].

[17] Since the writing of this paper, Dr. A. V. Balakrishnan has informed the author that he has proved the following long-standing conjecture concerning the general case of M equiprobable signals. If the dimensionality of the signal space (roughly $2TW$) is at least $M - 1$, then the signals, envisaged as points in signal space, should be placed at the vertices of an $(M - 1)$-dimensional regular simplex (*i.e.*, a polyhedron, each vertex of which is equally distant from every other vertex); in this situation, (44) holds for all i, and the right-hand side of (45) becomes $-E/(M - 1)$. The problem of a signal space of smaller dimensionality than $M - 1$ has not been solved in general.

[18] If the signals $s_i(t)$ are of finite duration, then their spectra extend over all f, and nonoverlapping bands are therefore not possible. An approximation is achieved by spacing the band centers by amounts large compared to the bandwidths.

The envelope of the output signal component of the kth matched filter is then

$$\left| 2 \sum_{l=1}^{L} e^{-j\theta_l} \int_0^\infty S_k^*(j2\pi f) S_i(j2\pi f) e^{j2\pi f(t-t_l)} \, df \right|. \tag{48}$$

Since the output envelopes of the matched filters are to be sampled at $t = t_r$ $(r = 1, \cdots, L)$, it seems reasonable to require that in the absence of noise these samples should all be zero for $k \neq i$. That is, (48) should be zero for all $k \neq i$ at all $t = t_r$. For lack of knowledge of the θ_L's, a sufficient condition for this to occur is that (46) be zero at all $t = t_r - t_l$.

An extension of this argument to the case of unknown modulation delays considered in connection with Fig. 6 leads similarly to a requirement that, for all $k \neq i$, (46) vanish over the whole interval $-A \leq t \leq A$, where $A = t_b - t_a$, t_a and t_b being the parameters referred to in Fig. 6. This may not be possible with physical signals, and some approximation must then be sought. Here is another unsolved problem.

A final desirable property of both radar and communication signals which is worth mentioning is one arising from the use of a peak-power limited transmitter. For such a transmitter, operation at rated average power often points to the use of a constant-amplitude signal, *i.e.*, one in which only the phase is modulated. A signal of this sort is also demanded by certain microwave devices.

Having thus closed parentheses on a rather lengthy detour into the problem of signal specification, let us recall the question on which we opened them. We had decided that the constraints of practical realizability had limited us to the consideration of filters which can, in fact, be built. Looking at any particular class of such filters—for example, those with less than 1000 lumped elements, with coils having Q's less than 200—we ask, "Which members of this class are matched to desirable signals?" We have gotten some idea of what constitutes a desirable signal. Let us now briefly examine a few proposed classes of filters and see to what extent this question has been answered for these classes.

VI. Some Forms of Matched Filters

It is not intended to give herein an exhaustive treatment of all solutions obtained to the problem of matched-filter realization; indeed, such a treatment would be neither possible nor desirable. We shall, rather, concentrate on three classes of solutions which seem to have attracted the greatest attention. Even in consideration of these we shall be brief, for details are adequately given elsewhere, in many cases in this issue.

A. Tapped-Delay-Line Filters

Let us first consider matched filters for the class of signals generatable as the impulse response of a filter of the form of Fig. 13. The spectrum of a signal of this class has the form

$$S(j2\pi f) = F(j2\pi f) \sum_{i=0}^{n} G_i(j2\pi f) e^{-j2\pi f \Delta_i}, \tag{49}$$

which is, of course, the transfer function of the filter of Fig. 13. It is immediately clear that a filter matched to this signal may be constructed in the form of Fig. 14, for the transfer function of the filter shown there is

$$\begin{aligned} H(j2\pi f) &= F^*(j2\pi f) \sum_{i=0}^{n} G_i^*(j2\pi f) e^{-j2\pi f(\Delta_n - \Delta_i)} \\ &= S^*(j2\pi f) e^{-j2\pi f \Delta_n}. \end{aligned} \tag{50}$$

The filters of Figs. 13 and 14 are thus candidates for the filter pair appearing in Fig. 7.

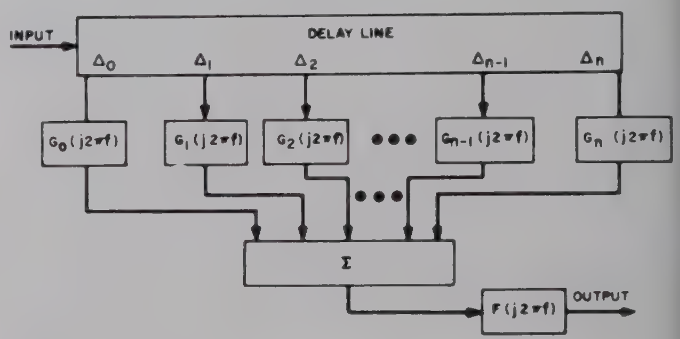


Fig. 13—A tapped-delay-line signal generator.

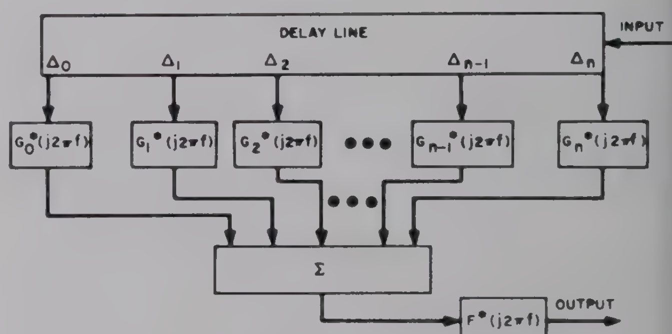


Fig. 14—A tapped-delay-line matched filter.

If $F(j2\pi f)$ and $G_i(j2\pi f)$ are assigned phase functions which are uniformly zero, then $F^*(j2\pi f) = F(j2\pi f)$ and $G_i^*(j2\pi f) = G_i(j2\pi f)$, and the two filters of Figs. 13 and 14 become identical except for the end of the delay line which is taken as the input. [The same identity of Figs. 13 and 14 is obviously also obtained, except for an unimportant discrepancy in delay, if $F(j2\pi f)$ and $G_i(j2\pi f)$ have linear phase functions, the slopes of all the latter being equal.] The advantages of having a single filter which can perform the tasks both of signal generation and signal processing are obvious, especially in situations such as radar, where the transmitter and receiver are physically at the same location.

Having defined a generic form of matched filter, we are still left with the problem of adjusting its characteristics [$F(j2\pi f)$, $G_i(j2\pi f)$ and Δ_i, all i] to correspond to a de-

irable signal. A possibility which immediately comes to nind is to set

$$\left.\begin{aligned} F(j2\pi f) &= \begin{cases} \dfrac{1}{2W}, & |f| \le W \\ 0, & |f| > W \end{cases} \\ G_i(j2\pi f) &= a_i \\ \Delta_i &= \frac{i}{2W} \end{aligned}\right\}. \tag{51}$$

'or, the signal which corresponds to such a choice has, rom (49), the spectrum

$$S(j2\pi f) = \begin{cases} \dfrac{1}{2W} \displaystyle\sum_{i=0}^{n} a_i e^{-j2\pi f(i/2W)}, & |f| \le W, \\ 0, & |f| > W, \end{cases} \tag{52}$$

nd the signal itself therefore has the form

$$s(t) = \sum_{i=0}^{n} a_i \frac{\sin \pi(2Wt - i)}{\pi(2Wt - i)}. \tag{53}$$

t is well known [43] that *any* signal limited to the band $|f| \le W$ can be represented in a form similar to (53), ut with the summation running over all values of i; he a_i's are in fact just the values of $s(t)$ at $t = i/2W$. n (53) we therefore have a band-limited low-pass signal or which

$$s\left(\frac{i}{2W}\right) = \begin{cases} a_i, & i = 0, \cdots, n \\ 0, & \text{other integral values of } i \end{cases}. \tag{54}$$

lthough $s(t)$ is not *uniformly* zero outside of $0 \le t \le$ $/2W$, it is seen from (53) and (54) that, at least for rge n, the duration of $s(t)$ is effectively $T = n/2W$. he time-bandwidth product of the signal, which we have ound to be a very important parameter, is thus approxi- ately $TW = n/2$. Notice that this is proportional to e number of taps on the delay line and the number f multipliers, a_i, which in this case justifies the use of e TW product as a measure of the complexity of the lter.

It would seem that we have here, in one swoop, solved e problem of signal specification and matched-filter esign, for we now have means available for obtaining th any desired band-limited signal and the filter matched it.[19] There are two drawbacks which mar this hopeful tlook, however, one practical and the other theoretical. he first is that, at least at present, we do not know how choose the a_i's so that $s(t)$ is desirable in the senses, y, of our discussions of (43) and (46). More basic is e fact that truly band-limited signals are not physically alizable; that is, the transfer function $F(j2\pi f)$ of (51) nnot be achieved, even theoretically. Using a realizable approximation to $F(j2\pi f)$ complicates the choice of the a_i's: for even if we were aware of how to choose the a_i's for the ideal $F(j2\pi f)$, it would not be clear how the use of an approximation would then affect the "desirability" of $s(t)$. This is not to say that the solution embodied in (51) should be discarded, but only that it must be further investigated to render it practically realizable.

Another possible choice of characteristics for the filters of Figs. 13 and 14 involves letting the pass bands of the filters $G_i(j2\pi f)$ be nonoverlapping, or essentially so [27]. [$F(j2\pi f)$ may here be considered to be unity, since limitation of the signal bandwidth is accomplished by the G_i's.] The purpose here is to afford independent control of various frequency bands of $S(j2\pi f)$ [cf. (49)], to the end of satisfying whatever requirements have been placed on the signal spectrum by constraints on (43) and (46). If we are considering only one signal which is not subject to doppler shift, then we are concerned only with constraints on (42), a special case of (43). In this case, control of various frequency ranges of $S(j2\pi f)$ is a direct method of achieving the energy-spectrum shaping mentioned in connection with (42); further, the use of a long enough delay line in conjunction with enough filters, G_i, will result in the desired large TW product.[20] For this special case, then, the use of a tapped delay line with nonoverlapping filters yields a possible desirable solution to our problem. More generally, however, when there are doppler shifts and/or many signals, we again must profess ignorance of how to select the G_i's and Δ_i's properly. Again further investigation is called for.

A third choice of characteristics for the filters of Figs. 13 and 14, *viz.*,

$$\left.\begin{aligned} F(j2\pi f) &= \frac{\sin \pi f \Delta}{\pi f} e^{-i\pi f\Delta} \\ G_i(j2\pi f) &= b_i = \pm 1 \\ \Delta_i &= i\,\Delta \end{aligned}\right\}, \tag{55}$$

has received considerable attention [12], [27], [44]. Note that the impulse response of $F(j2\pi f)$ is a rectangular pulse of unit height, and width Δ, starting at $t = 0$.[21] The impulse response of the filter of Fig. 13, *i.e.*, the matched signal, therefore has the form of Fig. 15—a low-pass sequence of positive and negative pulses, shown

[19] We have explicitly given only the low-pass case, in which, idendentally, the F and G's have the desirable zero phase functions. e band-pass case is obtainable by replacing the low-pass $F(j2\pi f)$ (51) with its band-pass equivalent, letting the a_i's be complex ., contain phase shifts), and letting $\Delta_i = i/W$, where W is the nd-pass band-width.

[20] If, in particular, the impulse responses of the filters G_i all have an effective duration of Δ, and the tap spacings in Fig. 13 are $\Delta_i - \Delta_{i-1} = \Delta$, then a "stepped-frequency" signal is obtained; that is, $s(t)$ is a continuous succession of nonoverlapping "pulses" of different frequencies. If, in addition, the bands of the G_i's are adjacent and have widths of the order of $1/\Delta$, the TW product for $s(t)$ and its matched filter is of the order of $n\Delta(n/\Delta) = n^2$. Here is a degenerate case in which we have generated a signal whose TW product is of the order of n^2 with a filter comprising a number of parameters proportional to n; that is, in this case TW is not a good measure of the filter's complexity, being much too large. The degeneracy involved here is not without its disastrous effects, however, as we shall see later when considering "chirp" signals.

[21] Needless to say, the $F(j2\pi f)$ can be eliminated in the signal-generator of Fig. 13, and the resulting filter driven by such a rectangular pulse, instead of an impulse. The $F(j2\pi f)$ is still required in the matched filter, however [40].

for $n = 10$. An equivalent band-pass case is obtained merely by using a band-pass equivalent of the $F(j2\pi f)$ in (55); the resulting signal is a sequence of sine-wave pulses of some frequency f_0, each pulse having either 0° or 180° phase. The search for a desirable signal is now simply the search for a desirable sequence, $b_0, b_1, \cdots, b_n$, of $+1$'s and -1's.

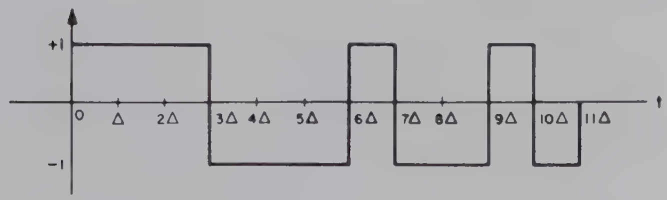


Fig. 15—A signal corresponding to (55).

Notice that $s(t)$ is a constant-amplitude signal, a property we have already seen to be important in some situations. For such signals, for the band-pass case implicit in (43), we easily may show that

$$| \chi(0, \phi) | = T \left| \frac{\sin \pi\phi T}{\pi\phi T} \right|, \tag{56}$$

where T is the signal duration, equal to $(n + 1)\,\Delta$ in Fig. 15. This is the cross section of the $| \chi(t, \phi) |$ surface in the ϕ direction; it has the shape shown in Fig. 16. In view of our discussion of the properties of $| \chi(t, \phi) |$, this shape seems reasonably acceptable: a central peak of the "minimum" possible "width," with no serious spurious peaks.

The cross section of $| \chi(t, \phi) |$ in the t direction—*i.e.*, $| \chi(t, 0) |$—can be made to have a desirable shape by proper choice of the sequence $b_0, b_1, \cdots, b_n$. Classes of sequences suitable from this point of view have, in fact, been found. The members of one such class [1], [47] have the desirable property that the central peak has a width of the order of Δ [the reciprocal of the "bandwidth" of $s(t)$] and is $(n + 1)$ times as high as any subsidiary peak. The sequence of Fig. 15, of length $n + 1 = 11$, is in fact a member of this class and has the $| \chi(t, 0) |$ shown in Fig. 17. Other members of the class with lengths 1, 2, 3, 4, 5, 7, and 13 have been found. Unfortunately, no odd-length sequences of length greater than 13 exist; it is a strong conjecture that no longer even-length sequences exist either.

If we for the moment redefine $| \chi(t, 0) |$ as the response envelope of the matched filter to the signal made periodic with period $(n + 1)\,\Delta$, we find that at least for the odd-length sequences we have just discussed, $| \chi(t, 0) |$ retains a desirable (albeit necessarily periodic) shape; for example, the new, "periodic" $| \chi(t, 0) |$ function for the sequence in Fig. 15 is shown in Fig. 18. This property is of interest in radar, where the signal is often repeated periodically or quasi-periodically.

Another class of binary sequences, $b_0, b_1, \cdots, b_n$, which have periodic $| \chi(t, 0) |$ functions of the desirable

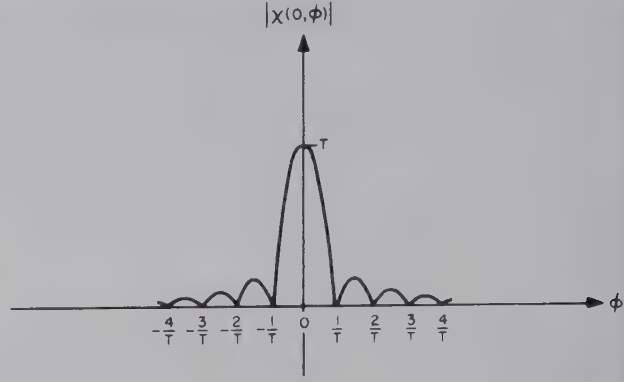


Fig. 16—The $|\chi(0, \phi)|$ function for the signal in Fig. 15.

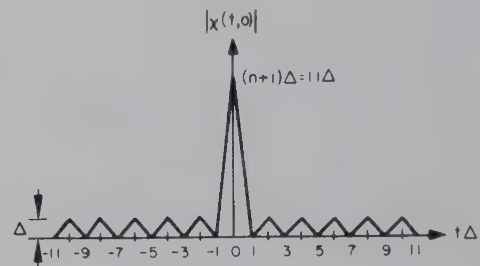


Fig. 17—The $|\chi(t, 0)|$ function for the signal in Fig. 15.

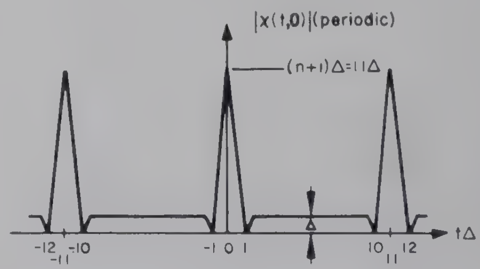


Fig. 18—The periodic $|\chi(t, 0)|$ function for the signal in Fig. 15.

form shown in Fig. 18 are the so-called linear maximal-length shift-register sequences. These have lengths $2^p - 1$, where p is an integer; the number of distinct sequences for any p is $\Phi(2^p - 1)/p$. where $\Phi(x)$, the Euler phi function, is the number of integers less than x which are prime to x [59]. The class of linear maximal-length shift-register sequences has been studied in great detail [10], [16], [17], [42], [44], [46], [59], [61]. In general, their aperiodic $| \chi(t, 0) |$ functions leave something to be desired [26].

Other classes of binary sequences with two-level periodic $| \chi(t, 0) |$ functions such as shown in Fig. 18 have been investigated [18], [60].

So far we have discussed the desirability of certain binary sequences for use in (55) only from the point of view of the form of the $| \chi(t, \phi) |$ function along the coordinate axes. No general solution has been given for the form of the function off the axes, but certain conjectures can be made, which have been roughly corroborated by trying special cases. It is felt [44], at least for long linear shift-register sequences, that the central peak

of $|\chi(t, \phi)|$ indeed has the minimum cross-sectional "area" of $1/TW$, and has subsidiary peaks in the plane of height no greater than the order of $1/\sqrt{n+1}$ of the height of the central peak [$(n+1)$ is the length of the sequence].

No general conclusions have been reached concerning the *joint* desirability, from the viewpoint of (46), of several of these binary codes.

The actual synthesis of matched filters with the characteristics of (55) is considered in detail elsewhere in this issue [24].

As a final example of a choice of characteristics of the filters of Figs. 13 and 14, let us consider

$$\left.\begin{aligned} F(j2\pi f) &= \frac{\sin \pi f a}{\pi f} e^{-i\pi f a} \\ G_i(j2\pi f) &= 1, \quad \text{all} \quad i \\ \Delta_i &= i\,\Delta, \qquad \Delta > a \end{aligned}\right\}. \tag{57}$$

The impulse response of the signal-generating filter in this case is a sequence of $(n+1)$ rectangular pulses of width a, starting at times $t = i\Delta$. (Such a signal, of course, is more easily generated by other means.) The ambiguity function for the band-pass analog of this signal, which is often used in radar systems, is rather undesirable, having pronounced peaks periodically both in the t and ϕ directions [44], [57], but the signal has the virtue of simplicity. The filter which is matched to the signal is often called a pulse integrator, since its effect is to add up the received signal pulses; alternatively, it is called a comb filter, since its transmission function, $|S(j2\pi f)|^2$, has the form of a comb with roughly (Δ/a) teeth of "width" $1/(n+1)\,\Delta$, spaced by $1/\Delta$ cps [14], [54].

B. Cascaded All-Pass Filters

Another way to get a signal-generating and matched filter pair is through the use of cascaded elementary all-pass networks [27], [48]. An elementary all-pass network is defined here as one which has a pole-zero configuration like that in Fig. 19. Such a network clearly has a transfer function which has constant magnitude at all frequencies, f, on the imaginary axis.

Now, suppose that many (say, n) elementary all-pass networks are cascaded, as in Fig. 20, to form an over-all network with the uniform pole-zero pattern of Fig. 21. The over-all transfer function, $G(j2\pi f)$, will have a constant magnitude; its phase will be approximately linear over some bandwidth W, as shown by the upper curve of Fig. 22.[22] If some of the elementary networks of Fig. 20 are now grouped together into a network, A, with transfer function $G_A(j2\pi f)$, and the rest into a network, B, with transfer function $G_B(j2\pi f)$, both $G_A(j2\pi f)$ and $G_B(j2\pi f)$ still have constant magnitudes, but neither necessarily will have a linear phase function; however, the two phase

[22] We are again considering the "low-pass" case for convenience. The "band-pass" analog is obvious.

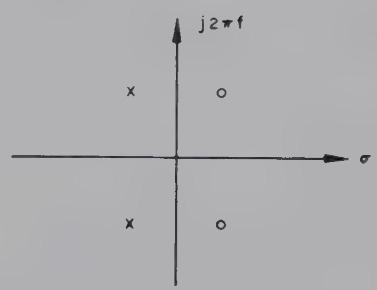


Fig. 19—The pole-zero configuration of an elementary all-pass network.

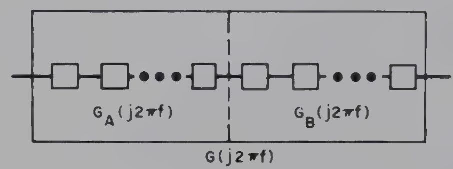


Fig. 20—A cascade of elementary all-pass networks.

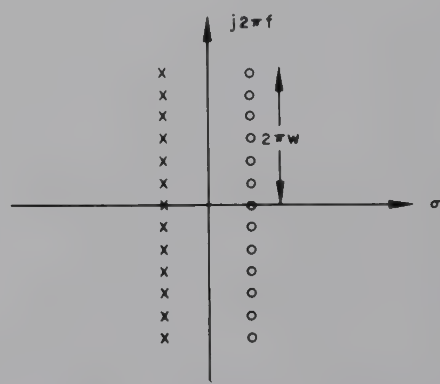


Fig. 21—The pole-zero configuration for the over-all network of Fig. 20.

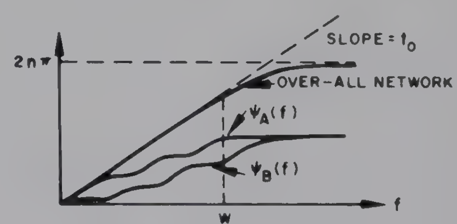


Fig. 22—Phase functions corresponding to the networks in Fig. 20.

functions must add to approximately a linear phase within the band $0 \leq f \leq W$ (see Fig. 22). *Within this band*, the relationship between the A and B transfer functions is

$$e^{-i\psi_A(f)} e^{-i\psi_B(f)} = e^{-i2\pi f t_0}, \tag{58}$$

where t_0 is the slope of the over-all phase function, and ψ_A and ψ_B are the phase functions of the A and B networks, respectively.

Suppose network A is driven by a pulse of bandwidth W, such as might be supplied by the pulse-forming network, $F(j2\pi f)$, of (57) ($a \approx 1/W$). Then the output

signal of network A will have the spectrum $S(j2\pi f) = F(j2\pi f)e^{-i\psi_A(f)}$. From (58), therefore, over the bandwidth of $F(j2\pi f)$,

$$F^*(j2\pi f)e^{-i\psi_B(f)} = F^*(j2\pi f)e^{i\psi_A(f)}e^{-i2\pi f t_0} = S^*(j2\pi f)e^{-i2\pi f t_0}. \quad (59)$$

We hence have, at least approximately, a filter pair $F(j2\pi f)\, G_A(j2\pi f)$ and $F^*(j2\pi f)\, G_B(j2\pi f)$ which are matched.

The function of the filter $F(j2\pi f)$ is merely that of band limitation; therefore F should be made as simple as possible, preferably with zero or linear phase. The problems of choosing $G_A(j2\pi f)$ so as to obtain a desirable signal, and of actually synthesizing the filters, are considered elsewhere in this issue [48].

C. Chirp Filters

Let us now consider, as a final class of signals, signals of the form

$$s(t) = A(t)\cos(\psi_0 + 2\pi f_0 t + 2\pi k t^2), \quad 0 \le t \le T, \quad (60)$$

where $A(t)$ is some slowly varying envelope. Such signals, whose frequency—more correctly, whose phase derivative—varies linearly with time, have appropriately been called "chirp" signals.

In general, the spectrum of a chirp signal has a very complicated form. For the case of a constant-amplitude pulse, $A(t)$ = constant, an exact expression has been given [4]; this indicates, as one might expect, that in practical cases the spectrum has approximately constant amplitude and linearly increasing group time delay of slope k over the band $f_0 \le f \le f_0 + kT$, and is approximately zero elsewhere. The corresponding matched filter then has approximately constant amplitude and linearly *decreasing* group time delay of slope $-k$ over the same band.

Chirp signals and the synthesis of their associated matched filters have been studied in great detail [3], [4], [6], [8], [22], [32], [44], [57]. The chief virtue of this class of signals is their ease of generation: a simple frequency-modulated oscillator will do. Similarly, chirp matched filters are relatively easy to synthesize. On the other hand, from the point of view of the ambiguity function, $|\chi(t, \phi)|$—*i.e.*, of radars in which doppler shift is important—a chirp signal is rather undesirable. The area in the (t, ϕ) plane over which the ambiguity function for a chirp signal with a gaussian envelope is large is shown in Fig. 23 [44], [57]. Along the t and ϕ axes the "width" of the $|\chi(t, \phi)|$ surface is satisfactorily small: $1/W$ and $1/T$, respectively. But the surface, instead of being concentrated around the origin as desired, is spread out along the line $\phi = kt$; in fact, the area of concentration is very much greater than the minimum value, $1/TW$. This is equivalent to saying that it is very hard to determine whether the chirp signal has been given a time delay t or a frequency shift kt, a fact which is obvious from an inspection of the signal waveform.

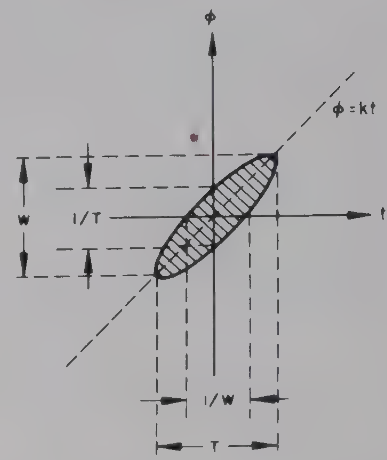


Fig. 23—The ambiguous "area" for a chirp signal.

VII. Conclusion

It must be admitted that, in regard to matched filters, the proverbial "state of the art" is somewhat less than satisfactory. We have seen that there are two basic problems to be solved simultaneously: the specification of a desirable signal or set of them, and the synthesis of their associated matched filters. In regard to the former, we have but the barest knowledge of the freedom with which we can constrain the "desirability" functions (43) and/or (46) and still expect physical signals; we know even less how to solve for the signals once both desirable and allowable constraints are set. It was partially for these reasons that we attacked a different problem: of a set of physical signals corresponding to a class of filters which we can build, which are the most desirable? In even this we were stopped, except for some special cases.

Nor can we assume that we are very sophisticated in the area of actually constructing filters which we say, on paper, we can build. It has become apparent that the TW product for a signal is a most important parameter; in general, the larger the product, the better the signal. But at the present time the synthesis of filters with TW products greater that several hundred may be deemed exceptional.

Clearly, much more effort is needed in this field. One hopes that the present issue of the Transactions will stimulate just such an effort.

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Joint Estimation of Delay, Doppler, and Doppler Rate*

PHILLIP BELLO†

Summary—**The methods of inverse probability have been used by Woodward and others to obtain lower bounds on variances of radar parameter estimators. Previous results on the lower bounds of variances of delay and Doppler estimators have assumed that the reflecting object travels with a constant line of sight velocity and does not cause scintillation in the radar return. Using the inverse probability approach, this paper derives expressions for the minimum variances of estimators of the delay, Doppler, and Doppler rate of a radar return assumed to consist of a long train of pulses with independent scintillation from pulse to pulse.**

I. Introduction

With the aid of inverse probability techniques, Woodward[1] derived an expression for the minimum error obtainable in the measurement of the delay of a radar return pulse from a stationary reflecting object. The interfering noise was assumed to be additive white Gaussian noise. Manasse[2] employed the same techniques to investigate the minimum errors in the joint estimation of the delay and Doppler of a radar return pulse immersed in additive white Gaussian noise. He assumed the reflecting object to be moving with a constant line of sight (los) velocity.

In an actual situation, the los velocity is generally time varying. This time variation will be unimportant in influencing delay and Doppler measurements if the observation interval[3] of the incoming data is sufficiently short. However, there are situations in practice where the observation interval is long. A notable example is the tracking radar wherein the range tracking loop may have an integration time of the order of a second.

In this paper, the methods of inverse probability are used, following Woodward,[1] to investigate the problem of the joint estimation of delay, Doppler, and Doppler rate. No attempt is made to include the effect of other radar parameters, such as azimuth and elevation, on the above estimation problem. The interference on the radar return is assumed to be of both a multiplicative and an additive nature. A narrow band complex Gaussian process multiplying the complex representation of the received signal is used to account for scintillation on the radar return. The additive noise is the usual receiver noise (assumed to be white Gaussian).

To simplify subsequent analysis, the complex representation of real waveforms will be used.[1,4,5,6] Also it will be assumed that the reader is familiar with the work of Woodward[1] in the area of statistical radar theory.

II. Model of Transmission Channel

It will be assumed that the operations that transform the transmitted (complex) signal into the received (complex) signal are represented in the block diagram of Fig. 1.

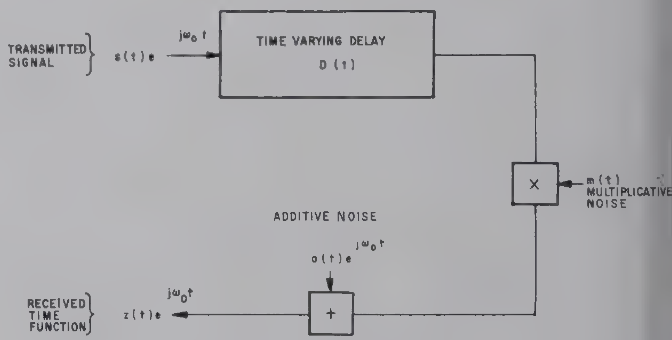

Fig. 1—Representation of transmission channel.

As discussed in the introduction, the complex representation of real waveforms will be used. Thus the complex transmitted signal $s(t)e^{j\omega_0 t}$, additive noise $a(t)e^{j\omega_0 t}$, and received waveform $z(t)e^{j\omega_0 t}$ have nonzero power spectra only for positive frequencies. It is assumed moreover that these waveforms have their energy concentrated about the "carrier" frequency ω_0. In this case the quantities $s(t)$, $a(t)$, and $z(t)$ are complex low frequency waveforms appropriately called "complex envelopes" since their magnitudes equal the envelopes and their angles equal the phases of the corresponding narrow band signals. The complex function $m(t)$ introduces amplitude and phase scintillation on the received signal. The low frequency noise functions $a(t)$ and $m(t)$ are assumed to be independent, complex, valued, normally distributed processes. Their primary difference lies in the fact that the bandwidth

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† Applied Research Lab., Sylvania Electronic Systems, Waltham, Mass.

[1] P. M. Woodward, "Probability and Information Theory," McGraw-Hill Book Co. Inc., New York, N. Y.; 1953.

[2] R. Manasse, "Range and Velocity Accuracy from Radar Measurements," M. I. T. Lincoln Lab., Lexington, Mass., Group Rept. 312–26; February, 1955.

[3] The term "observation interval" is defined strictly as the time interval during which the received waveform (noise combined with signal) is observed. The values of the received waveform on the observation interval represent the input data from which estimates of radar parameters are made. For a practical system the observation interval may be regarded as the approximate time interval that the received waveform is manipulated, integrated, or operated upon to obtain an estimate of radar parameters.

[4] J. Dugundji, "Envelopes and pre-envelopes of real waveforms," IRE Trans. on Information Theory, vol. IT-4, pp. 53–5 March, 1958.

[5] R. Arens, "Complex processes for envelopes of normal noise," IRE Trans. on Information Theory, vol. IT-3, pp. 204–20 September, 1957.

[6] D. Gabor, "Theory of communications," *J. IEE*, pt. III, v 93, pp. 429–457; 1946.

of $a(t)$ is much larger than that of $s(t)$, while the bandwidth of $m(t)$ is very much less than that of $s(t)$.

The time varying delay line is included in Fig. 1 to account for the time varying los round trip range $2R(t)$. Specifically, $D(t)$ is equal to the delay suffered by an impulse which is received at time t. This delay is twice the time it took the transmitted impulse to reach the target. Thus,

$$\frac{D}{2} = \frac{1}{c} R\left(t - \frac{D}{2}\right); \qquad c = \text{velocity of light}, \tag{1}$$

since the one way range traversed is the range $D/2$ seconds before the impulse is received. If $D(t)$ is lnown as a function of time, $R(t)$ may be computed with the aid of (1). This paper will be confined exclusively to the problem of estimating $D(t)$, or rather estimating a set of discrete parameters which are assumed to define $D(t)$. Specifically, it will be assumed that $D(t)$ may be represented by the quadratic:

$$D(t) = \tau + \frac{\nu}{f_0} t + \frac{\alpha}{2f_0} t^2; \qquad -\frac{T_0}{2} \leq t \leq \frac{T_0}{2}, \tag{2}$$

over the observation interval $-T_0/2 \leq t \leq T_0/2$. The quantity τ is the delay suffered by a narow pulse which is received at $t = 0$. Similarly ν and α are the Doppler shift and rate of change of Doppler shift for the same narrow pulse. The carrier frequency is $f_0 = \omega_0/2\pi$.

It will be assumed that the time varying delay line of Fig. 1 is a passive lossless device. On this basis a delay line input $f(t)$ yields an output pulse $\sqrt{1 - \dot{D}(t)} f[t - D(t)]$ where $\dot{D} = dD/dt$. The square root factor keeps the output pulse energy equal to the input pulse energy. Actually,

$$\dot{D} = \frac{dD}{dt} <<< 1,$$

since $\dot{D}$ is of the order of twice the ratio of the los velocity of the reflecting object to the velocity of light. Thus the square root factor will be negligibly different from one. However, the square root factor is included since certain later derivations are thereby simplified. Inspection of Fig. 1 then shows that the complex envelope $z(t)$ of the received waveform is given by

$$z(t) = a(t) + m(t)s[t - D(t)]e^{-i\omega_0 D(t)} \sqrt{1 - \dot{D}(t)}. \tag{3}$$

The carrier frequency ω_0 is assumed known. Consequently, the complex envelope of the received time function contains as much information about the parameters τ, ν, α as the actual received time function.

It is clear that knowledge of τ, ν, and α is equivalent to knowledge of $D(t)$ for $-T_0/2 \leq t \leq T_0/2$. We will assume that there is defined a joint parameter probability density function (pdf) $W_0(\tau, \nu, \alpha)$ which indicates the state of *a priori* knowledge concerning the parameters τ, ν, α. The pdf $W_0(\tau, \nu, \alpha)$ is called the *a priori* parameter pdf. Once we have observed the received waveform, our knowledge of the radar parameters changes. Our new state of knowledge concerning the radar parameters is characterized by the *a posteriori* pdf $W(\tau, \nu, \alpha/z(t))$. This is the conditional parameter pdf, given the fact that the received waveform (complex envelope) $z(t)$ has been observed for $-T_0/2 \leq t \leq T_0/2$.

Before proceeding with the presentation of results, some basic assumptions that have been made will be indicated. First it is assumed that the transmitted waveform $s(t)$ consists of a periodic repetition of identical pulses and that the number of pulses included in an observation interval is much greater than one. It is assumed that successive radar pulses are far enough apart relative to the "correlation time" of the scintillation so that the scintillation can be assumed independent on a pulse to pulse basis. Furthermore, it is assumed that individual pulses are short enough so that the scintillation is essentially constant over a pulse width. This means that the effect of scintillation on an individual pulse is an amplitude change and a phase shift.

III. A Posteriori Parameter Density Function

The additive noise $a(t)$ is assumed to have a power spectrum which is constant at $4N_0$ over $-W/2 \leq f \leq W/2$ and zero elsewhere. In this way $\text{Re}\{a(t) \exp[j\omega_0 t]\}$ is constant at N_0 over $W/2 \leq |f - f_0| \leq W/2$. Also $m(t)s(t - D)e^{i\omega_0 D}$ is assumed to be band-limited to less than $W/2$ cycles per second. Then $z(t)$ is band-limited to the interval $-W/2 < f < W/2$. On this basis $z(t)$ may be represented by a complex vector $\bar{z}$ in a space of WT_0 dimensions whose coordinates are samples of $z(t)$ taken every $1/W$ seconds over the observation interval. It is desired to determine $W(\bar{p}/\bar{z})$, the *a posteriori* parameter density function. The parameter vector $\bar{p}$ has been used to denote the parameter triple (τ, ν, α). Once $W(\bar{p}/\bar{z})$ is obtained, minimum variance estimates of (τ, ν, α) may be found. Using Baye's Rule, it follows that

$$W(\bar{p}/\bar{z}) = \frac{W_0(\bar{p})}{W_z(\bar{z})} W(\bar{z}/\bar{p}), \tag{3}$$

where $W_z(\bar{z})$ is the marginal pdf of received sample values and $W(\bar{z}/\bar{p})$ is the conditional pdf of received sample values, given the fact that the radar parameters are $\bar{p} = (\tau, \nu, \alpha)$. The function $W(\bar{z}/\bar{p})$ is called the likelihood function when it is considered as a function of τ, ν, α. Once $W(\bar{z}/\bar{p})$ is evaluated, $W(\bar{p}/\bar{z})$ can be found from (3) since $W_0(\bar{p})$ and $W_z(\bar{z})$ are known.

For convenience let

$$q(t) = s[t - D(t)]e^{-i\omega_0 D(t)} \sqrt{1 - \dot{D}(t)} \tag{4}$$

be defined as the received signal in the absence of additive and multiplicative noise. Then

$$\bar{z} = \bar{a} + \bar{m}\bar{q}, \tag{5}$$

where $\bar{a}$, $\bar{m}$, $\bar{q}$ are complex WT_0-dimensional vectors whose coordinates are sample values of $a(t)$, $m(t)$, and $q(t)$, respectively, taken every $1/W$ seconds. The pdf $W(\bar{z}/\bar{p})$ will be evaluated by the multiple integration,

$$W(\bar{z}/\bar{p}) = \int W(\bar{z}, \bar{m}/\bar{p})\, d\bar{m} = \int W(\bar{z}/\bar{m}, \bar{p}) W_m(\bar{m})\, d\bar{m}$$
$$= E_{\bar{m}}[W(\bar{z}/\bar{m}, \bar{p})], \tag{6}$$

where $W(\bar{z}, \bar{m}/\bar{p})$ is the conditional joint pdf of $\bar{z}$, $\bar{m}$ given $\bar{p}$, $W(\bar{z}/\bar{m}, p)$ is the conditional pdf of $\bar{z}$ given $\bar{m}$, $\bar{p}$, and $W_m(\bar{m})$ is the pdf of $\bar{m}$. $E_\theta [f(\theta)]$ is the expectation of $f(\theta)$ over θ.

According to the discussion in Section II it is assumed that the transmitted waveform is a sequence of pulses and that the scintillation on successive pulses is independent. For this reason (and because of the independence of the additive noise from pulse to pulse) it is possible to factor the conditional density function $W(\bar{z}/\bar{p})$ into the form

$$W(\bar{z}/\bar{p}) = \prod_{-r}^{r} P_k(\bar{z}_k/\bar{p}), \tag{7}$$

where $\bar{z}_k$ is a WT_1-dimensional complex vector whose coordinates consist of the set of samples of $z(t)$ contained within the kth radar repetition period (of duration T_1). For convenience, the number of radar periods in T_0 is taken to be the odd integer

$$n = 2r + 1 = \frac{T_0}{T_1}. \tag{8}$$

The pdf $P_k(\bar{z}_k/\bar{p})$ is determined from

$$P_k(\bar{z}_k/\bar{p}) = \int W(\bar{z}_k/m_k, \bar{p}) W_{m_k}(m_k)\, dm_k$$
$$= E_{m_k}[W(\bar{z}_k/m_k, \bar{p})]. \tag{9}$$

Since the scintillation is assumed constant over a pulse width, m_k is a one-dimensional normal complex variate with density function

$$W_{m_k}(m_k) = \frac{1}{2\pi\sigma_m^2} \exp\left[\frac{-|m_k|^2}{2\sigma_m^2}\right], \tag{10}$$

where

$$\sigma_m^2 = \tfrac{1}{2}E_{m_k}(|m_k|^2). \tag{11}$$

When $\bar{p}$ is known then $q(t)$ is known. It follows from (5) that

$$W(\bar{z}_k/m_k, \bar{p}) = W_{\bar{a}_k}(\bar{z}_k - m_k q_k) \tag{12}$$

where the pdf of $\bar{a}_k$, the complex WT_1-dimensional vector representing $a(t)$[7] in the kth radar period, is given by

$$W_{\bar{a}_k} = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{2WT_1} \exp\left[-\frac{1}{4N_0}\int_k |a(t)|^2\, dt\right];$$
$$2WT_1 \gg 1, \tag{13}$$

where $\int_k$ indicates an integration over the time interval of the kth radar period and $\sigma^2 = 2WN_0$. It follows that

$$W(\bar{z}_k/m_k, \bar{p}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{2WT_1}$$
$$\cdot \exp\left[-\frac{1}{4N_0}\int_k |z(t) - m_k q(t)|^2\, dt\right]. \tag{14}$$

Without difficulty, one may evaluate the expectation

$$E_{m_k}[W(\bar{z}_k/m_k, \bar{p})]$$

as

$$P_k(\bar{z}_k/\bar{p}) = E_{m_k}[W(\bar{z}_k/m_k, \bar{p})] = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{2WT_1} \frac{1}{2\sigma_m^2 c}$$
$$\cdot \exp\left[-\frac{1}{4N_0}|z(t)|^2\, dt\right] \exp\left[\frac{\left|\int_k z^*(t) q(t)\, dt\right|^2}{16N_0^2 c}\right], \tag{15}$$

where

$$c = \frac{1}{2\sigma_m^2} + \frac{\int_k |q(t)|^2\, dt}{4N_0}. \tag{16}$$

The average signal energy e_p received for the kth radar pulse is given by

$$e_p = E_{m_k}\left\{\tfrac{1}{2}\int_k |m_k|^2 \, |s(t - D)e^{-j\omega_0 D}\sqrt{1 - \dot{D}}|^2\, dt\right\}$$
$$= \sigma_m^2 \int_k |s(t)|^2\, dt. \tag{17}$$

According to the assumptions of Section II, $s(t)$ consists of a succession of identical pulses repeated at the radar repetition rate, *i.e.*,

$$s(t) = \sum_{-r}^{r} \mu(t - kT_1). \tag{18}$$

It will be assumed for convenience that

$$\int_{-\infty}^{\infty} |\mu(t)^2|\, dt = 1. \tag{19}$$

It will be further assumed that each radar repetition period defined for the received waveform contains one complete return pulse. Then

$$\int_k |s(t)|^2\, dt = 1 \tag{20}$$

and

$$e_p = \sigma_m^2. \tag{21}$$

If the average received energy ratio per pulse is defined as

$$\rho_p = \frac{e_p}{N_0}, \tag{22}$$

then

$$c = \frac{1}{4e_p}[2 + \rho_p]. \tag{23}$$

[7] Woodward, *op. cit.*, Section 4.9. Note that the N_0 used in this paper is $\frac{1}{2}$ that used by Woodward.

Using definitions (13)–(15),

$$P_k(\bar{z}_k/\bar{p}) = \left(\frac{1}{\sigma\sqrt{2}}\right)^{2WT_1} \exp\left[-\frac{1}{4N_0}\int_k |z|^2\,dt\right]\left(\frac{2}{2+\rho_p}\right) \cdot \exp\left[\frac{\rho_p}{4N_0(2+\rho_p)}|X_k|^2\right], \quad (24)$$

where the correlation integral X_k is given by

$$X_k = \int_k z^*(t)s[t - D(t)]e^{-i\omega_0 D(t)}\sqrt{1-\dot{D}}\,dt. \quad (25)$$

It is not difficult to show that $|X_k|^2$ can be interpreted as an appropriate "matched filtering" operation followed by envelope detection. However, this physical interpretation will not be discussed further since the primary purpose of this paper is the determination of lower bounds on the variances of estimators of (τ, ν, α).

Substituting $P_k(\bar{z}_k/\bar{p})$ of (24) into (7) yields

$$W(\bar{z}/\bar{p}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{2WT_0}\left(\frac{2}{2+\rho_p}\right)^n \cdot \exp\left[-\frac{1}{4N_0}\int_{T_0}|z|^2\,dt\right] \cdot \exp\left[\frac{\rho_p}{4N_0(2+\rho_p)}\sum_{-r}^{r}|X_k|^2\right]. \quad (26)$$

Returning now to (3) it becomes clear that $W(\tau, \nu, \alpha/\bar{z})$ may be written as

$$W(\tau, \nu, \alpha/\bar{z}) = KW_0(\tau, \nu, \alpha) \cdot \exp\left[\frac{\rho_p}{4N_0(2+\rho_p)}A(\tau, \nu, \alpha)\right], \quad (27)$$

where

$$A(\tau, \nu, \alpha) = \sum_{-r}^{r}|X_k|^2 = \sum_{-r}^{r}\left|\int_k z^*(t)s(t-D)e^{-i\omega_0 D}\sqrt{1-\dot{D}}\,dt\right|^2; \quad (28)$$

and K can be determined from the fact that

$$\iiint W(\tau, \nu, \alpha/\bar{z})\,d\tau\,d\nu\,d\alpha = 1. \quad (29)$$

It is clear that any information in the received waveform about the radar parameters of the target can be derived once $A(\tau, \nu, \alpha)$ is known. No additional information about the observed waveform is required. Thus, following Woodward,[8] a receiver which computes $A(\tau, \nu, \alpha)$ may be called a "sufficient" receiver.

IV. Estimation of Radar Parameters

In this Section, consideration will be given to the problem of estimating the radar parameters τ_a, ν_a, α_a, of a reflecting object a whose radar return is present in the received waveform $z(t)$. This paper will be concerned with obtaining minimum variance estimates of the parameters. It is well known that the estimator which yields minimum variance estimates is one that computes the mean values of the parameters on the basis of the *a posteriori* density function of these parameters. Thus if $\hat{\tau}_a$ is an estimate of τ_a, it will have minimum variance if selected so that

$$\hat{\tau}_a = \bar{\tau} = \iiint \tau W(\tau, \nu, \alpha/\bar{z})\,d\tau\,d\nu\,d\alpha. \quad (30)$$

Exactly analogous statements apply to estimates ν_a, α_a. Although (30) and its analogs will yield minimum variance estimates, they are not practical to instrument. A more practical estimator is the maximum likelihood estimator. This method estimates τ_a, ν_a and α_a by finding the coordinates of the maximum of the likelihood function $W(\bar{z}/\tau, \nu, \alpha)$. Note, however, that if the following two conditions are satisfied:

1) the *a priori* parameter pdf $W_0(\tau, \nu, \alpha)$ is not sharply peaked or at least is slowly varying compared to the *a posteriori* pdf $W(\tau, \nu, \alpha/\bar{z})$;
2) the likelihood function $W(z/\tau, \nu, \alpha)$ has a center of symmetry at which its maximum is located;

then the maximum likelihood estimate is very nearly equal to the minimum rms error estimate. The satisfaction of conditions 1) and 2) lead to the maximum likelihood estimate for the following reasons. First we note that the *a posteriori* pdf is given by

$$W(\tau, \nu, \alpha \mid \bar{z}) = KW_0(\tau, \nu, \alpha)W(\bar{z}/\tau, \nu, \alpha). \quad (31)$$

If condition 1) is satisfied we may write

$$W(\tau, \nu, \alpha \mid \bar{z}) \sim W(\bar{z} \mid \tau, \nu, \alpha). \quad (32)$$

If in addition condition 2) is satisfied, the *a posteriori* density function has a center of symmetry. However, the center of gravity of a distribution is located at the center of symmetry, if such a center of symmetry exists. But in view of (32) and the fact that the likelihood function has a center of symmetry at which its maximum is located, it follows that the coordinates of the maximum of the likelihood function are minimum rms error estimates of τ_a, ν_a, α_a.

Condition 1) will be assumed valid in this paper. It will be demonstrated now that for sufficiently large received energy and for transmitted pulses satisfying certain regularity conditions, condition 2) above will be satisfied. Then, subject to these conditions, one may obtain minimum variance estimates by using the maximum likelihood estimator.

Assuming the validity of condition 1) above,

$$W(\tau, \nu, \alpha/\bar{z}) = K\exp\left\{\frac{\rho_p}{4N_0(2+\rho_p)}A(\tau, \nu, \alpha)\right\} \quad (33)$$

where K is a proportionality constant. $A(\tau, \nu, \alpha)$ will now be separated into the sum of a signal function $S(\tau, \nu, \alpha)$ and a noise function $N(\tau, \nu, \alpha)$. To perform this separation, note that the received time function $z(t)$ is given by

[8] Woodward, *op. cit.*, p. 67.

$$z(t) = a(t) + m(t)s[t - D_a(t)]e^{-j\omega_0 D_a(t)} \sqrt{1 - \dot{D}_a}, \quad (34)$$

where

$$D_a(t) = \tau_a + \frac{\nu_a}{f_0} t + \frac{\alpha_a}{2f_0} t^2; \qquad -\frac{T_0}{2} < t < \frac{T_0}{2} \quad (35)$$

is the delay variation caused by target a. For convenience define

$$q_a(t) = s[t - D_a(t)]e^{-j\omega_0 D_a(t)} \sqrt{1 - \dot{D}_a(t)}. \quad (36)$$

Then X_k may be separated into the sum of two terms, one of which involves the additive noise and the other the received signal

$$X_k = S_k + N_k, \quad (37)$$

where

$$S_k = m_k^* \int_k q_a^*(t)q(t)\, dt,$$

$$N_k = \int_k a^*(t)q(t)\, dt. \quad (38)$$

Note that N_k, like $a(t)$, is a complex normally distributed process. In terms of S_k and N_k,

$$S(\tau, \nu, \alpha) = \sum_{-r}^{r} | S_k |^2,$$

$$N(\tau, \nu, \alpha) = \sum_{-r}^{r} 2 \operatorname{Re} \{S_k N_k^*\} + \sum_{-r}^{r} | N_k |^2$$

$$= \overline{(N\tau, \nu, \alpha)} + F(\tau, \nu, \alpha), \quad (39)$$

where $\overline{N(\tau, \nu, \alpha)}$ is the average value of $N(\tau, \nu, \alpha)$ and $F(\tau, \nu, \alpha)$ is its fluctuating value. Although not explicitly noted, it should be realized that $S(\tau, \nu, \alpha)$ and $N(\tau, \nu, \alpha)$ are functions of τ_a, ν_a, α_a.

By using the Schwartz inequality it is readily determined that

$$\text{Max } [S(\tau, \nu, \alpha)] = S(\tau_a, \nu_a, \alpha_a)$$

$$= \sum_{-r}^{r} | m_k |^2 = 2E, \quad (40)$$

where E is the total energy in the received pulse train over the observation interval. Thus the coordinates of the maximum value of the signal function are just the true values of the radar parameters.

Assuming that the bandwidth W of the additive noise is much larger than the signal pulse bandwidth, it is readily shown that

$$\overline{N(\tau, \nu, \alpha)} = 4nN_0$$

$$\overline{F^2(\tau, \nu, \alpha)} = 16N_0^2\left[n + \frac{E}{N_0}\right], \quad (41)$$

where $\overline{F^2(\tau, \nu, \alpha)}$ is the average of F^2 over the additive noise alone. The discussion of this paper will be confined to the practical situation in which the rms value of the fluctuations in the noise function are small compared to the maximum value of the signal function, $S(\tau_a, \nu_a, \alpha_a) = 2E$. Thus, it is assumed that

$$\sqrt{\overline{F^2}} = 4N_0\sqrt{n + \frac{E}{N_0}} \ll 2E. \quad (42)$$

Let the total received energy ratio be defined as

$$\mathcal{R} = \frac{E}{N_0}. \quad (43)$$

Then inequality (42) may be rewritten as

$$\frac{\sqrt{\overline{F^2}}}{2E} = \frac{2\sqrt{n + \mathcal{R}}}{\mathcal{R}} \ll 1. \quad (44)$$

Having separated $A(\tau, \nu, \alpha)$ into the sum of a signal function and a noise function, we obtain

$$W(\tau, \nu, \alpha/\bar{z}) = K \exp\left\{\frac{\rho_p}{4N_0(2 + \rho_p)}\left[S(\tau, \nu, \alpha) + F(\tau, \nu, \alpha)\right]\right\}. \quad (45)$$

[K in (45) differs from that in (33).] Assuming that the energy ratio $\mathcal{R}$ is sufficiently large to satisfy inequality (44), $S(\tau, \nu, \alpha)$ will (with high probability) swamp $F(\tau, \nu, \alpha)$ in the vicinity of $\tau = \tau_a$, $\nu = \nu_a$, $\alpha = \alpha_a$. It is appropriate, then, to consider the contribution of $S(\tau, \nu, \alpha)$ alone to the *a posteriori* pdf by defining

$$W_s(\tau, \nu, \alpha/\bar{z}) = K \exp\left\{\frac{\rho_p}{2(2 + \rho_p)} \cdot \frac{S(\tau, \nu, \alpha)}{2N_0}\right\}. \quad (46)$$

It is clear that the exponential term in $W_s(\tau, \nu, \alpha/\bar{z})$ has its maximum at $\tau = \tau_a$, $\nu = \nu_a$, $\alpha = \alpha_a$, and that this maximum is

$$\exp\left\{\frac{\rho_p}{2(2 + \rho_p)} \mathcal{R}\right\}. \quad (47)$$

This maximum increases exponentially with the energy ratio $\mathcal{R}$. Since the total "volume" under the *a-posteriori* pdf is fixed at unity, an exponential increase of $W_s(\tau, \nu, \alpha/\bar{z})$ in the vicinity of its maximum value must be followed by a corresponding decrease in the size of $W_s(\tau, \nu, \alpha/\bar{z})$ away from its maximum value. In the limit as ρ tends to infinity, the *a posteriori* density function must approach a unit impulse located at $\tau = \tau_a$, $\nu = \nu_a$ and $\alpha = \alpha_a$. For values of $\mathcal{R}$ such that

$$\mathcal{R} \gg 2\left[1 + \frac{2}{\rho_p}\right], \quad (48)$$

the exponential term in $W_s(\tau, \nu, \alpha/\bar{z})$ will be a sharply "peaked" function of τ, ν, α in the vicinity of τ_a, ν_a, α_a. It will be assumed that $\mathcal{R}$ is sufficiently large to satisfy inequality (48). When the number of radar pulses received is much larger than one, satisfaction of inequality (44) will usually imply satisfaction of inequality (48) unless the average energy ratio per pulse, ρ_p, is unusually low.

Since this paper is concerned only with the case wherein the *a posteriori* pdf is sharply peaked about its maximum value (which maximum value is moreover, located in the

vicinity of τ_a, ν_a, α_a), it is only necessary, in investigating the nature of $W(\tau, \nu, \alpha/\bar{z})$, to examine $S(\tau, \nu, \alpha)$ in the vicinity of $(\tau_a, \nu_a, \alpha_a)$. Presuming the existence of a Taylor series expansion of $S(\tau, \nu, \alpha)$ about τ_a, ν_a, α_a (which implies certain regularity conditions on the transmitted waveform), it must be that

$$\frac{\rho_p}{2(2+\rho_p)} \cdot \frac{S(\tau, \nu, \alpha)}{2N_0} = \frac{\mathcal{R}\rho_p}{2(2+\rho_p)} - \tfrac{1}{2}Q(\tau', \nu', \alpha') + \text{higher order terms}, \tag{49}$$

where $Q(\tau', \nu', \alpha')$ is a positive definite quadratic form in $\tau' = \tau - \tau_a$, $\nu' = \nu - \nu_a$, $\alpha' = \alpha - \alpha_a$. The reason for this restricted form of Taylor series about τ_a, ν_a, α_a is that $S(\tau, \nu, \alpha)$ has a positive maximum at $\tau = \tau_a$, $\nu = \nu_a$, $\alpha = \alpha_a$. Retaining no terms higher than second order, then

$$W_s(\tau, \nu, \alpha/\bar{z}) \approx K \exp\{-\tfrac{1}{2}Q(\tau', \nu', \alpha')\}, \tag{50}$$

which is in the form of a three-dimensional normal pdf. Thus $\tau = \tau_a$, $\nu = \nu_a$, and $\alpha = \alpha_a$ is a point of symmetry in the pdf $W_s(\tau, \nu, \alpha/\bar{z})$ of (50).

As long as inequalities (44) and (48) are satisfied, the shape of the *a posteriori* pdf $W(\tau, \nu, \alpha/\bar{z})$ cannot differ to any extent from that of $W_s(\tau, \nu, \alpha/\bar{z})$. It is not difficult to demonstrate that the first order effect of $F(\tau, \nu, \alpha)$ in modifying $W_s(\tau, \nu, \alpha/z)$ is a shift in the location of its maximum value by a random amount. This statement assumes the existence of a representation of $W_s(\tau, \nu, \alpha/\bar{z})$ of the form of (50). The proof involves the inclusion of only the linear terms in a Taylor series expansion of $F(\tau, \nu, \alpha)$ about τ_a, ν_a, α_a. The linear terms may then be interpreted as shifts in the quadratic form variables of $W_s(\tau, \nu, \alpha/\bar{z})$. Thus, for sufficiently high energy ratio [and assuming the validity of (49)], one may write

$$W(\tau, \nu, \alpha/\bar{z}) \approx K \exp\{-\tfrac{1}{2}Q(\tau' + \epsilon_\tau, \nu' + \epsilon_\nu, \alpha' + \epsilon_\alpha)\}, \tag{51}$$

where ϵ_τ, ϵ_ν, and ϵ_α are random variables representing the random displacement from τ_a, ν_a, α_a of the location of the maximum value of $W(\tau, \nu, \alpha/\bar{z})$. Thus (under appropriate restrictions) the validity of condition 2) above has been demonstrated.

It is worthwhile to discuss briefly the validity of inequalities (44) and (48). Taken together they specify that $\mathcal{R}$ must be greater than some value, say $\mathcal{R}_{\min}$. Since $\mathcal{R}$ is a random variable, such an inequality can be satisfied only with a certain probability. In fact since

$$\mathcal{R} = \frac{1}{2N_0} \sum_{-r}^{r} |m_k|^2, \tag{52}$$

it is (apart from a scale change) distributed according to the chi-squared pdf with $2n$ degrees of freedom.[9] One may then compute the probability with which $\mathcal{R}$ fails to exceed $\mathcal{R}_{\min}$.

[9] H. Cramer, "Mathematical Methods of Statistics," Princeton University Press, Princeton, N. J., p. 233; 1954.

However, it is not difficult to argue that if reliable signal detection is to take place, then $\mathcal{R} \gg \mathcal{R}_{\min}$. The heuristic argument here is simply that a necessary condition leading to reliable detection is that the *a posteriori* probability density function must have a well-defined peak located in the vicinity of the true signal parameter value; and from our previous discussion, this is essentially equivalent to the condition $\mathcal{R} \gg \mathcal{R}_{\min}$.

Thus any expressions derived in the following Section for lower bounds on errors in estimation of delay, Doppler, and Doppler rate should be regarded as bounds relevant to a class of receivers which have effected reliable signal detection.

V. Error Variances

Since the minimum variance estimator is one that computes the mean value of the parameters on the basis of the *a posteriori* density function [see (51)] it follows that

$$\sigma_\tau^2 = \int_{-\infty}^{\infty} W_z(\bar{z}) \iiint_{-\infty}^{\infty} (\tau - \bar{\tau})^2 W(\tau, \nu, \alpha/\bar{z})\, d\tau\, d\alpha\, d\bar{z} \tag{53}$$

is the value of the minimum error variance in estimation of τ_a with analogous expressions for σ_ν^2, σ_α^2, the minimum error variances in estimation of ν_a and α_a, respectively. As (53) indicates, σ_τ^2 may be computed by first evaluating

$$\hat{\sigma}_\tau^2 = \iiint_{-\infty}^{\infty} (\tau - \bar{\tau})^2 W(\tau, \nu, \alpha/\bar{z})\, d\tau\, d\nu\, d\alpha, \tag{54}$$

the conditional error variance, and then averaging over all possible received waveforms. Therefore,

$$\sigma_\tau^2 = \overline{\hat{\sigma}_\tau^2}, \qquad \sigma_\nu^2 = \overline{\hat{\sigma}_\nu^2}, \qquad \sigma_\alpha^2 = \overline{\hat{\sigma}_\alpha^2}, \tag{55}$$

where the overline indicates an average with respect to the received waveform.

On the assumption that (51) represents $W(\tau, \nu, \alpha/\bar{z})$, it is readily determined that

$$\sigma_\tau^2 = \iiint \tau^2 \exp[-\tfrac{1}{2}Q(\tau, \nu, \alpha)]\, d\tau\, d\nu\, d\alpha \tag{56}$$

with similar expressions for σ_ν^2, σ_α^2. The positive definite quadratic form $Q(\tau, \nu, \alpha)$ may be expressed as the matrix product

$$Q(\tau, \nu, \alpha) = [\tau\ \nu\ \alpha] M \begin{bmatrix} \tau \\ \nu \\ \alpha \end{bmatrix}, \tag{57}$$

where the positive definite matrix M is given by

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}. \tag{58}$$

From the well-known properties of the Gaussian multivariate density function one can write

$$\hat{\sigma}_\tau^2 = \frac{|M_{11}|}{|M|}, \qquad \hat{\sigma}_\nu^2 = \frac{|M_{22}|}{|M|}, \qquad \hat{\sigma}_\alpha^2 = \frac{|M_{33}|}{|M|}, \tag{59}$$

where $|M|$ is the determinant of M and $|M_{ii}|$ is the minor of the ij element in M.

Since the m_{jk} have certain common factors, it is convenient to define the matrix R,

$$R = pM = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}, \tag{60}$$

where

$$p = \frac{2(2 + \rho_p)}{n\rho_p^2}. \tag{61}$$

Then

$$\frac{|M_{ij}|}{|M|} = p\frac{|R_{ij}|}{|R|}. \tag{62}$$

The expressions in (59) assume that all three parameters τ_a, ν_a, α_a are unknown *a priori*. If one of these is known, say α_a, then the appropriate quadratic form is found by setting $\alpha = 0$ in $Q(\tau, \nu, \alpha)$ and carrying out integrations only over τ and ν. Then one readily determines that

$$\begin{aligned} \hat{\sigma}_\tau^2 &= p\frac{r_{22}}{|R_{33}|}, & \hat{\sigma}_\nu^2 &= p\frac{r_{11}}{|R_{33}|}; & \alpha_a &\ \text{known}, \\ \hat{\sigma}_\tau^2 &= p\frac{r_{33}}{|R_{22}|}, & \hat{\sigma}_\alpha^2 &= p\frac{r_{11}}{|R_{22}|}; & \nu_a &\ \text{known}, \\ \hat{\sigma}_\nu^2 &= p\frac{r_{33}}{|R_{11}|}, & \hat{\sigma}_\alpha^2 &= p\frac{r_{22}}{|R_{11}|}; & \tau_a &\ \text{known}. \end{aligned} \tag{63}$$

Similarly when two parameters are known,

$$\begin{aligned} \sigma_\tau^2 &= p\frac{1}{r_{11}}; & \nu_a, \alpha_a &\ \text{known}, \\ \sigma_\nu^2 &= p\frac{1}{r_{22}}; & \tau_a, \alpha_a &\ \text{known}, \\ \sigma_\alpha^2 &= p\frac{1}{r_{33}}; & \tau_a, \nu_a &\ \text{known}. \end{aligned} \tag{64}$$

In Appendix I it is demonstrated that

$$\begin{aligned} r_{11} &= G_{20}A_0, \\ r_{22} &= G_{20}\,\delta^2 A_2 + 2G_{11}\,\delta A_1 + G_{02}A_0, \\ r_{33} &= T_0^2\left[G_{20}\frac{\delta^2}{4}A_4 + G_{11}\,\delta A_3 + G_{02}A_2\right], \\ r_{12} &= G_{20}\,\delta A_1 + G_{11}A_0, \\ r_{13} &= T_0\left[G_{20}\frac{\delta}{2}A_2 + G_{11}A_1\right], \\ r_{23} &= T_0\left[G_{20}\frac{\delta^2}{2}A_3 + G_{11}\frac{3}{2}\,\delta A_2 + G_{02}A_1\right], \end{aligned} \tag{65}$$

where A_q is a random variable given by

$$A_q = \frac{1}{2e_p n^{q+1}}\sum_{k=-r}^{r} k^q\,|m_k|^2, \tag{66}$$

and the G_{20}, G_{11}, G_{02} are coefficients in a Taylor series expansion of a function $G(\xi, \eta)$ as indicated below:

$$\begin{aligned} G(\xi, \eta) &= \left(1 - \frac{\xi}{2f_0}\right)\left|\int \mu^*(t)\mu\left[\left(1 - \frac{\xi}{2f_0}\right)t - \frac{\eta\xi}{2f_0} + \eta\right]e^{-j2\pi\xi t}\,dt\right|^2 \\ &= 1 - G_{20}\frac{\eta^2}{2} - G_{11}\xi\eta - G_{02}\frac{\xi^2}{2} + \cdots. \end{aligned} \tag{67}$$

The remaining symbol, δ, in (65) is given by

$$\delta = \frac{T_0}{f_0}. \tag{68}$$

To compute the error variances, σ_τ^2, σ_ν^2 and σ_α^2, it is necessary to average $\hat{\sigma}_\tau^2$, $\hat{\sigma}_\nu^2$, $\hat{\sigma}_\alpha^2$ with respect to the $|m_k|^2$. In the most general case this involves computation of the average of $|R_{ij}|/|R|$. This average appears to be difficult if at all possible to carry through analytically. Some approximation must then be resorted to. This paper will only be concerned with the situation in which the number of received radar pulses n is much greater than one. Consider the mean and variance of A_q for n large

$$\bar{A}_q = \frac{1}{n^{q+1}}\sum_{k=-r}^{r} k^q = \begin{cases} 0; & n \ \text{odd} \\ \dfrac{1}{2^q(q+1)}; & n \ \text{even} \end{cases} \tag{69}$$

$$\sigma_{A_q}^2 = \frac{1}{n^{2(q+1)}}\sum_{k=-r}^{r} k^{2q} = \frac{1}{2^{2q}(2q+1)}\cdot\frac{1}{n}.$$

Thus the mean value of A_q is independent of n while the variance decreases by $1/n$. As n increases, A_q may be regarded as approaching a deterministic quantity $\bar{A}_q$. Similarly, r_{jk} may also be regarded as approaching $\bar{r}_{jk}$ for large n. If the approximation

$$R = \bar{R}; \qquad n \ \text{large} \tag{70}$$

is used, one obtains as error variances (in the 3-unknown parameter case):

$$\begin{aligned} \sigma_\tau^2 &= p\frac{|\bar{R}_{11}|}{|\bar{R}|}; & \sigma_\nu^2 &= \frac{|\bar{R}_{22}|}{|\bar{R}|}; \\ \sigma_\alpha^2 &= p\frac{|\bar{R}_{33}|}{|\bar{R}|}; & n &\ \text{large}, \end{aligned} \tag{71}$$

where $|\bar{R}_{jj}|$ is the minor of the jj element in $\bar{R}$ and $|\bar{R}|$ is the determinant of $\bar{R}$. Even if n is not large, however, there is reason to believe that the variances in (71) represent lower bounds on the error variances. This assertion is based on the author's belief that the diagonal elements of the inverse of a positive definite matrix are convex functions of the matrix elements. If $W(x_1, x_2, \cdots$

) is a convex function of $x_1, x_2, \cdots, x_n$, then it is well
iown[10] that

$$\overline{W(x_1, x_2 \cdots x_n)} \geq W(\bar{x}_1, \cdots \bar{x}_n). \tag{72}$$

hus, if $\hat{\sigma}_\tau^2$, $\hat{\sigma}_\nu^2$, and $\hat{\sigma}_\alpha^2$ are convex functions of the r_{ii}, it
llows that

$$\sigma_i^2 = \overline{\hat{\sigma}_i^2} \geq p \frac{|\bar{R}_{ii}|}{|R|}, \tag{73}$$

here $\sigma_1^2 = \sigma_\tau^2$, $\sigma_2^2 = \sigma_\nu^2$, $\sigma_3^2 = \sigma_\alpha^2$. Actually the author has
ıly been able to determine that the diagonal elements
the inverse of a positive definite matrix have the above
nvex property for a 1×1 and 2×2 matrix. To de-
rmine by direct evaluation whether this property
ists for a 3×3 matrix offers a prohibitive amount of
lculation. It appears that a general proof for an $n \times n$
atrix should be constructible; however, the author has
en unable to do so.

In one particular case the actual value of $\sigma^2 = \overline{\hat{\sigma}^2}$ may
evaluated. This is the case in which ν_a, α_a are known
d it is desired to measure τ_a. From (64) and (65),

$$\hat{\sigma}_\tau^2 = \frac{2pn}{G_{20}} \cdot \frac{1}{\chi_n}; \qquad \nu_a, \alpha_a \text{ known}, \tag{74}$$

ıere χ_n is a chi-squared pdf with $2n$ degrees of freedom,

$$\chi_n = \sum_{k=r}^{r} \frac{|m_k|^2}{e_p}. \tag{75}$$

is readily determined that

$$\overline{\frac{1}{\chi_n}} = \frac{1}{2(n-1)}. \tag{76}$$

ıerefore,

$$\sigma_\tau^2 = \frac{p}{G_{20}} \cdot \frac{n}{n-1} \to \frac{p}{G_{20}} \quad \text{for } n \text{ large}, \tag{77}$$

ıich, for large n, equals the value of σ_τ^2 that would be
mputed by replacing R by $\bar{R}$.

The elements of $\bar{R}$ are listed below:

$$\begin{aligned}
&= G_{20}; && \bar{r}_{12} = G_{11}, \\
&= G_{20}\frac{\delta^2}{12} + G_{02}; && \bar{r}_{13} = T_0 G_{20} \frac{\delta}{24}, \\
&= T_0^2\left[G_{20}\frac{\delta^2}{320} + G_{02}\frac{1}{12}\right]; && \bar{r}_{23} = T_0 G_{11} \frac{\delta}{8}.
\end{aligned} \tag{78}$$

is important to note that the characteristics of the
ınsmitted waveform enter the $\bar{r}_{ik}$, and thus enter the
pressions for error variance only through the coefficients
, G_{11}, and G_{02}.

The following Section will be concerned with a dis-
ssion of the nature of the coefficients G_{11}, G_{20}, and G_{02}.

VI. The Coefficients G_{20}, G_{11}, G_{02}

By expanding $G(\xi, \eta)$ in a Taylor series about $\xi = 0$, $\eta = 0$, it may be determined that

$$\frac{G_{20}}{2} = -C_{02}; \qquad G_{11} = 4\pi \,\text{Im}\,\{C_{11}\},$$

$$\frac{G_{02}}{2} = (2\pi)^2 C_{20} + \frac{2\pi \,\text{Im}\,\{C_{21}\}}{f_0} + \frac{1 - \text{Re}\,\{C_{22}\} - |C_{11}|^2}{4f_0^2}, \tag{79}$$

where

$$C_{mn} = \int_{-\infty}^{\infty} t^m \mu^*(t) \frac{d^n \mu(t)}{dt^n} \, dt. \tag{80}$$

If the spectrum of $\mu(t)$ is defined as

$$U(f) = \int_{-\infty}^{\infty} \mu(t) e^{-j2\pi f t} \, dt, \tag{81}$$

then an equivalent frequency domain expression for C_{mn} is

$$C_{mn} = (2\pi j)^{n-m} \int_{-\infty}^{\infty} f^n U(f) \frac{d^m U^*(f)}{df^m} \, df. \tag{82}$$

It is interesting to note that for $m = n$, the time and frequency domain integrals become identical in form. In evaluating the expressions of (79) it has been assumed, for convenience, that the "average" location of the pulse $\mu(t)$ in time and frequency is zero. Specifically, it is assumed that

$$\begin{aligned}
\int f \, |U(f)|^2 \, df &= 0, \\
\int t \, |\mu(t)|^2 \, dt &= 0.
\end{aligned} \tag{83}$$

The first integral in (83) is made zero by proper choice of carrier frequency, while the second is made zero by proper selection of time origin. It is readily seen that the expressions in (83) imply $C_{10} = C_{01} = 0$.

The coefficients C_{02} and C_{20} have simple interpretations,

$$\begin{aligned}
C_{02} &= -4\pi^2 F^2 = -4\pi^2 \int f^2 \, |U(f)|^2 \, df, \\
C_{20} &= \Delta^2 = \int t^2 \, |\mu(t)|^2 \, dt,
\end{aligned} \tag{84}$$

where F may be interpreted as the rms bandwidth and Δ the rms duration or pulse width of $\mu(t)$.

The coefficient G_{11} has appeared in the statistical radar theory literature before[11,12] but without any simple interpretation. A simple interpretation of G_{11} and some

[10] D. A. S. Fraser, "Nonparametric Methods in Statistics," n Wiley and Sons, Inc., New York, N. Y., p. 54; 1957.

[11] Manasse, *op. cit.*, p. 9.

[12] I. S. Reed, E. J. Kelley, and W. L. Root, "The Detection of Radar Errors in Noise, Part II: The Accuracy of Radar Measurements," M. I. T. Lincoln Lab., Lexington, Mass., Tech. Rept. No. 159; July 19, 1957.

other coefficients may be found by using the polar representation of $\mu(t)$,

$$\mu(t) = a_1(t)e^{j\phi(t)}. \tag{85}$$

For the practical situation in which the transmitted pulse has a bandwidth small compared to f_0, $a_1(t)$ is the envelope and $\phi(t)$ is the phase modulation of the transmitted pulse. Then it is readily determined that

$$G_{11} = 4\pi \operatorname{Im} \{C_{11}\} = 8\pi^2 C \tag{86}$$

where the parameter

$$C = \int t\left(\frac{\dot{\phi}}{2\pi}\right)a_1^2(t)\, dt; \qquad \dot{\phi} = \frac{d\phi}{dt} \tag{87}$$

may be interpreted as just the product of t by the instantaneous pulse FM, $\dot{\phi}/2\pi$, averaged with respect to a distribution of t values given by $a_1^2(t)$, the squared pulse envelope. It then becomes clear that C and thus G_{11} will vanish for a symmetrical pulse envelope whenever the pulse FM is an even function about the point of symmetry. Of course C will vanish for any pulse envelope if $\dot{\phi} = 0$, *i.e.*, if there is no FM.

In terms of $a_1(t)$ and $\phi(t)$, it is readily determined that

$$F^2 = F_\phi^2 + F_{a_1}^2 \tag{88}$$

where

$$F_{\dot{\phi}}^2 = \int_{-\infty}^{\infty} \left(\frac{\dot{\phi}}{2\pi}\right)a_1^2(t)\, dt;$$

$$F_{a_1}^2 = \int_{-\infty}^{\infty} \left(\frac{\dot{a}_1}{2\pi}\right)^2 dt; \qquad \dot{a}_1 = \frac{da_1}{dt} \tag{89}$$

Note that the mean squared bandwidth is the sum of two positive quantities. The first, F_ϕ^2, is zero only when the pulse FM vanishes and may be regarded as that component of the pulse mean squared bandwidth which is due to the pulse FM. Note that $F_{\dot{\phi}}^2$ has the interpretation of the mean squared value of the pulse FM when averaged with respect to the distribution $a_1^2(t)$. The second term $F_{a_1}^2$ is independent of the pulse FM and thus may be regarded as that portion of the pulse mean squared bandwidth which is due to the pulse envelope. By application of the Schwartz inequality, one readily determines that C has an upper bound $F_{\dot{\phi}}^2\Delta^2$, *i.e.*,

$$C^2 \leq F_{\dot{\phi}}\, \Delta^2. \tag{90}$$

When the pulse FM is sufficiently large, F^2 will be determined primarily by $F_{\dot{\phi}}^2$. Then C gives an interesting lower bound to the time-bandwidth product of a pulse.

The other coefficients may also be expressed in terms of $a(t)$ and $\phi(t)$. Thus it may be shown that

$$2\pi \operatorname{Im} \{C_{21}\} = 4\pi^2 \int t^2\left(\frac{\dot{\phi}}{2\pi}\right)a_1^2\, dt. \tag{91}$$

It will be convenient to define

$$\begin{aligned} K_{mn} &= \int t^m\left(\frac{\dot{\phi}}{2\pi}\right)^n a_1^2(t)\, dt, \\ L_m &= \int t^m\left(\frac{\dot{a}_1}{2\pi}\right)^2 dt. \end{aligned} \tag{92}$$

Then it is readily shown that

$$1 - \operatorname{Re} \{C_{22}\} - | C_{11} |^2 = 4\pi^2[K_{22} - C^2] + 4\pi^2 L_2 - \tfrac{1}{4}$$

$$2\pi \operatorname{Im} \{C_{21}\} = 4\pi^2 K_{21}\,.$$

Application of the Schwartz inequality yields various inequalities relating coefficients, such as

$$\begin{aligned} &C^2 \leq K_{22} \leq \sqrt{K_{40}K_{04}}\,, \\ &4\pi^2 L_2 \geq \tfrac{1}{4}, \\ &K_{21} \leq \Delta \sqrt{K_{22}}\,. \end{aligned} \tag{93}$$

The definition

$$\frac{G_{02}}{2} = 4\pi^2\, \Delta_1^2 \tag{94}$$

will be used, where

$$\Delta_1^2 = \Delta^2 + \frac{K_{21}}{f_0} + \frac{K_{22} - C^2 + L_2 - 1/16\pi^2}{4f_0^2}. \tag{95}$$

Because of the fact that $G(\xi, \eta)$ has its maximum at $G(\xi, \eta) = 1$, it follows that $G_{20}G_{02} - G_{11}^2 \leq 0$. This leads directly to

$$F^2\, \Delta_1^2 > C^2. \tag{96}$$

VI. Expressions for Error Variances

To obtain the error variance for large n, the diagonal elements of $[\bar{R}]^{-1}$ must be found. The matrix $\bar{R}$ is given by

$$\bar{R} = 8\pi^2 \begin{bmatrix} F^2 & C & T_0F^2\,\dfrac{\delta}{24} \\ C & F^2\,\dfrac{\delta^2}{12} + \Delta_1^2 & T_0C\,\dfrac{\delta}{8} \\ T_0F^2\,\dfrac{\delta}{24} & T_0C\,\dfrac{\delta}{8} & T_0^2\left[F^2\,\dfrac{\delta^2}{320} + \dfrac{\Delta_1^2}{12}\right] \end{bmatrix}. \tag{97}$$

After some algebraic manipulation, one finds (τ_a, ν_a, c unknown) that the error variances may be expressed in the forms

$$\sigma_\tau^2 = \frac{p}{8\pi^2 F^2}\, \frac{\left(\dfrac{b}{12} + 1\right)\left(\dfrac{3}{80}\, b + 1\right) - \dfrac{3}{16}\,\gamma}{\left(\dfrac{b}{12} + 1\right)\left(\dfrac{b}{60} + 1\right) - \left(1 + \dfrac{b}{10}\right)\gamma}\,,$$

$$\sigma_\nu^2 = \frac{p}{8\pi^2 \Delta_1^2}\, \frac{\dfrac{b}{60} + 1}{\left(\dfrac{b}{12} + 1\right)\left(\dfrac{b}{60} + 1\right) - \left(1 + \dfrac{b}{10}\right)\gamma}\,, \tag{98}$$

$$\sigma_\alpha^2 = \frac{3p}{2\pi^2\Delta_1^2 T_0^2}\frac{\left(\frac{b}{12}+1\right)-\gamma}{\left(\frac{b}{12}+1\right)\left(\frac{b}{60}+1\right)-\left(1+\frac{b}{10}\right)\gamma},$$

vhere the dimensionless quantities b, γ are given by

$$b = \frac{F^2}{\Delta_1^2}\,\delta^2 = \frac{F^2}{f_0^2}\cdot\frac{T_0^2}{\Delta_1^2},$$
$$\gamma = \frac{C^2}{F^2\Delta_1^2} < 1. \tag{99}$$

The expressions for the variances in (98) are somewhat nvolved. Considerable simplification results when $\gamma = 0$. 'he condition $\gamma = 0$ is, in fact, a desirable condition since t may be shown that increasing γ from zero only increases he error variances. From the definition of γ it becomes lear that $\gamma = 0$ implies $C = 0$ and vice versa (assuming F, Δ, finite). The conditions necessary on the transmitted vaveform to make $C = 0$ have been discussed in Section 'I [See (87)]. For $\gamma = 0$, the error variances become

$$\left.\begin{aligned}
\sigma_\tau^2 &= \frac{p}{8\pi^2F^2}\left[\frac{\frac{3}{80}b+1}{\frac{1}{60}b+1}\right]\\
\sigma_\nu^2 &= \frac{p}{8\pi^2\Delta_1^2}\frac{1}{\left(\frac{1}{12}b+1\right)} = \frac{p}{8\pi^2}\left[\frac{1}{\frac{\delta^2F^2}{12}+\Delta_1^2}\right]\\
\sigma_\alpha^2 &= \frac{3p}{2\pi^2\Delta_1^2T_0^2}\frac{1}{\left(\frac{1}{60}b+1\right)} = \frac{3p}{2\pi^2T_0^2}\left[\frac{1}{\frac{\delta^2F^2}{60}+\Delta_1^2}\right]
\end{aligned}\right\} C = 0. \tag{100}$$

Examination of the expression for σ_τ^2 shows that the ns error in measurement of τ_a is inversely proportional the reciprocal bandwidth of pulse times a factor inolving b. Since this factor varies monotonically from at $b = 0$ to 1.5 at $b = \infty$ it is of no importance in etermining σ_τ^2. Thus σ_τ^2 is essentially a function of only he bandwidth of the pulse.

Examination of (100) shows that σ_ν^2 and σ_α^2 are deendent upon both F^2 and Δ_1^2. For $C = 0$, Δ_1^2 is given by

$$\begin{aligned}
\Delta_1^2 &= \int t^2a_1^2(t)\,dt + \frac{1}{f_0}\int t^2\left(\frac{\dot\phi}{2\pi}\right)a_1^2\,dt\\
&\quad + \frac{1}{4f_0^2}\int t^2\left(\frac{\dot\phi}{2\pi}\right)^2a_1^2\,dt + \frac{1}{f_0^2}\left[\int t^2\left(\frac{\dot a_1}{2\pi}\right)^2dt - \frac{1}{16\pi^2}\right] \qquad (101)\\
&= \int t^2\left[1+\frac{\dot\phi}{2\pi}/2f_0\right]^2a_1^2(t)\,dt\\
&\quad + \frac{1}{f_0^2}\left[\int t^2\left(\frac{\dot a_1}{2\pi}\right)^2dt - \frac{1}{16\pi^2}\right].
\end{aligned}$$

For a narrow-band pulse,

$$\frac{\dot\phi}{2\pi} \ll f_0, \tag{102}$$

i.e., the peak FM is much less than the carrier frequency. Then

$$\int t^2\left[1+\frac{\dot\phi}{2\pi}/2f_0\right]^2a_1^2(t)\,dt \approx \int t^2a_1^2(t)\,dt = \Delta^2. \tag{103}$$

In order for the transmitted pulse to be narrow band, a necessary requirement is seen to be that the pulse width be much greater than the duration of an RF cycle, *i.e.*, that

$$\Delta \gg \frac{1}{f_0}. \tag{104}$$

It follows that the term $1/(4\pi f_0)^2$ in (101) may be neglected in comparison to Δ^2. Using approximations (103) and (104),

$$\Delta_1^2 = \Delta^2 + \frac{1}{f_0^2}\int t^2\left(\frac{a}{2\pi}\right)^2dt. \tag{105}$$

The error variances σ_ν^2 and σ_α^2 show an interesting behaviour as a function of transmitted pulse width. Assume that the width of $\mu(t)$ is varied without changing its shape, *i.e.*, consider a contraction or expansion of the time scale of $\mu(t)$. Let

$$\mu(t) = \frac{1}{\sqrt{k}}\,\mu'\left(\frac{t}{\sqrt{k}}\right), \tag{106}$$

where $\mu'(t)$ is a version of $\mu(t)$ with some normalized duration. The $\sqrt{k}$ is necessary to normalize $|\mu(t)|^2$ to unit area. If the transformation (106) is used, then it is readily demonstrated that

$$\begin{aligned}
\Delta^2 &= k^2(\Delta')^2,\\
F^2 &= \frac{1}{k^2}(F')^2, \qquad (107)\\
L_2 &= \int t^2\left(\frac{\dot a_1}{2\pi}\right)^2dt = \int t^2\left(\frac{\dot a_1^1}{2\pi}\right)^2dt = L_2',
\end{aligned}$$

i.e., the pulse width is increased by k, the bandwidth divided by k, and L_2 remains unchanged. The expressions for σ_ν^2 and σ_α^2 become

$$\begin{aligned}
\sigma_\nu^2 &= \frac{p}{8\pi^2}\left[\frac{1}{k^2(\Delta')^2+\frac{L_2}{f_0^2}+\frac{\delta^2(F')^2}{12k^2}}\right],\\
\sigma_\alpha^2 &= \frac{3p}{2\pi^2T_0^2}\left[\frac{1}{k^2(\Delta')^2+\frac{L_2}{f_0^2}+\frac{\delta^2(F')^2}{60k^2}}\right].
\end{aligned} \tag{108}$$

For small pulse widths, the term in $1/k^2$ becomes all important so that

$$\left.\begin{aligned} \sigma_\tau^2 &= \frac{p}{8\pi^2}\cdot\frac{9k^2}{4(F')^2} = \frac{p}{8\pi^2}\cdot\frac{9}{4F^2} \\ \sigma_\nu^2 &= \frac{p}{8\pi^2}\cdot\frac{12f_0^2k^2}{(F')^2T_0^2} = \frac{p}{8\pi^2}\cdot\frac{12f_0^2}{F^2T_0^2} \\ \sigma_\alpha^2 &= \frac{3p}{2\pi^2T_0^2}\cdot\frac{60f_0^2k^2}{(F')^2T_0^2} = \frac{3p}{2\pi^2}\cdot\frac{60f_0^2}{F^2T_0^4} \end{aligned}\right\}. \quad \begin{array}{l}\text{small pulse widths} \\ k^2 \to 0.\end{array} \tag{109}$$

For large pulse widths, the term in k^2 becomes all important so that

$$\left.\begin{aligned} \sigma_\tau^2 &\to \frac{p}{8\pi^2}\cdot\frac{k^2}{(F')^2} = \frac{p}{8\pi^2F^2} \\ \sigma_\nu^2 &\to \frac{p}{8\pi^2}\cdot\frac{1}{k^2(\Delta')^2} = \frac{p}{8\pi^2\Delta^2} \\ \sigma_\alpha^2 &\to \frac{3p}{2\pi^2T_0^2}\cdot\frac{1}{k^2(\Delta')^2} = \frac{3p}{2\pi^2T_0^2\Delta^2} \end{aligned}\right\} \begin{array}{l}\text{large pulse widths} \\ k^2 \to \infty.\end{array} \tag{110}$$

Thus, the error in measurement of Doppler and Doppler rate approaches zero not only as the pulse width gets very large but also as the pulse width gets very small. There is, in fact, a value of k, k_ν, at which the Doppler error is worst and a value of k, k_α at which the Doppler rate error is worst. These are

$$k_\nu^2 = \frac{b'}{\sqrt{12}}, \qquad b' = \frac{\delta F'}{\Delta'}. \qquad k_\alpha^2 = \frac{b'}{\sqrt{60}}, \tag{111}$$

The fact that the Doppler measurement error decreases monotonically as $1/\Delta$ for large pulse widths is to be expected from previous work on the measurement of Doppler on a single pulse. The fact that the Doppler and Doppler measurement error approach zero monotonically for small pulse widths might also have been expected from the following reasoning. When the pulse width is small one may obtain excellent delay measurement capability on a single pulse (although not good Doppler measurement capability). Thus one may operate upon delay information to obtain estimates of Doppler and Doppler rate with good results. Since the delay measurement error approaches zero as $1/F$ so also will the Doppler and Doppler rate measurement error.

An interesting corroboration of this reasoning may be found in the work of Manasse[13] who considered the problem of fitting a second degree polynomial to a set of independent equally-spaced measurements with equal variance. Thus, assume that a measurement of delay is made on each radar pulse, and from the set of n delay measurements, it is desired to estimate the second degree polynomial

$$D_a(t) = \tau_a + \frac{\nu_a}{f_0}\,t + \frac{\alpha_a}{2f_0}\,t^2.$$

From Manasse's results one readily deduces the following expressions for minimum error variance in measurement of τ_a, ν_a, α_a:

$$\begin{aligned} \sigma_\tau^2 &= \frac{9\sigma_0^2}{4n}, \\ \sigma_\nu^2 &= \frac{12\sigma_0^2f_0^2}{nT_0^2}, \\ \sigma_\alpha^2 &= \frac{720\sigma_0^2f_0^2}{nT_0^4}, \end{aligned} \tag{112}$$

where σ_0^2 is the variance of the measurement of delay on a single pulse. A comparison of (112) with (109) shows that if σ_0^2 is chosen as

$$\sigma_0^2 = \frac{pn}{8\pi^2F^2} = \left(\frac{2+\rho_p}{\rho_p}\right)\frac{1}{\rho_p(2\pi F)^2}, \tag{113}$$

then the expressions for error variances in (112) become identical to those in (109).

It is interesting to note that for ρ_p sufficiently large, τ_0^2 becomes identical in form to the expression for minimum error variance in delay measurement derived by Woodward for the case of a constant amplitude pulse. The expressions become identical (for ρ_p large), if ρ_p, the average received pulse energy ratio, is equated to the actual received pulse energy ratio in the constant amplitude pulse case.

Appendix I

Evaluation of r_{ik}

Let the Taylor series expansion of $S(\tau, \nu, \alpha)$ about $\tau' = 0$, $\nu' = 0$, $\alpha' = 0$ be written in the form

$$S(\tau, \nu, \alpha) = 2E - \tfrac{1}{2}Q'(\tau', \nu', \alpha') + (\text{higher order terms}), \tag{114}$$

where $Q'(\tau', \nu', \alpha')$ is a positive definite quadratic form in τ', ν', α'. Then from (49),

$$Q(\tau', \nu', \alpha') = \frac{1}{2e_pn}\cdot\frac{Q'(\tau', \nu', \alpha')}{p}. \tag{115}$$

In view of (60) and (115), the coefficients r_{ik} are just the coefficients in the quadratic form Q' divided by $2e_pn$. Since the function $S(\tau, \nu, \alpha)$ is given by

$$S(\tau, \nu, \alpha) = \sum_{s=-r}^{r} |m_s|^2 \left| \int_s q_a^*(t)q(t)\,dt \right|^2, \tag{116}$$

[13] R. Manasse, "Parameter Estimation Theory and Some Applications of the Theory to Radar Measurements," Mitre Corp. Tech. Rept., to be published.

ne may obtain the desired Taylor expansion of $S(\tau, \nu, \alpha)$ rom that of the function $R_s(\tau, \nu, \alpha)$ given by

$$R_s(\tau, \nu, \alpha) = \left| \int_s q_a^*(t) q(t)\, dt \right|^2. \tag{117}$$

When $\tau' = \nu = \alpha' = 0$, $q_a(t) = q(t)$. By application of he Schwartz inequality, it is readily determined that

$$\text{Max}\ \{R_s(\tau, \nu, \alpha)\} = R_s(\tau_a, \nu_a, \alpha_a) = 1. \tag{118}$$

Consequently (assuming certain regularity conditions on he transmitted waveform), $R(\tau, \nu, \alpha)$ has an expansion f the form

$$R_s(\tau, \nu, \alpha) = 1 - \tfrac{1}{2} Q_s(\tau', \nu', \alpha') + \cdots. \tag{119}$$

f the coefficients in $Q_s(\tau', \nu', \alpha')$ are denoted by $r_{jk}^{(s)}$, hen it is clear that

$$r_{jk} = \frac{1}{2e_p n} \sum_{s=-r}^{r} r_{jk}^{(s)}. \tag{120}$$

t remains then to determine the $r_{jk}^{(s)}$.

With the aid of (18) one may express $R_s(\tau, \nu, \alpha)$ in the rm

$$R_s(\tau, \nu, \alpha) = \left| \int_{-T_1/2}^{T_1/2} f_a^*(t, s) f(t, s)\, dt \right|^2, \tag{121}$$

here

$$\begin{aligned} f_a(t, s) &= \sqrt{1 - \dot{D}_a(t + sT_1)} \\ &\quad \cdot \mu[t - D_a(t + sT_1)] e^{-i\omega_0 D_a(t+sT_1)}, \\ f(t, s) &= \sqrt{1 - \dot{D}(t + sT_1)} \\ &\quad \cdot \mu[t - D(t + sT_1)] e^{-i\omega_0 D(t+sT_1)}, \end{aligned} \tag{122}$$

nd the first radar period is assumed to cover the time nterval $-T_1/2 < t < T_1/2$. While the delay terms $D_a(t)$ and $D(t)$ are quadratic functions of time over the bservation interval due to the presence of a rate of hange of Doppler, one may assume that a radar period is sufficiently short so that a linear approximation to D and D_a is valid over any given radar period. Thus the linear approximation

$$\begin{aligned} D(t + sT_1) &= D(sT_1) + t\dot{D}(sT_1) \\ &= \left[\tau + \frac{\nu}{f_0} sT_1 + \frac{\alpha}{2f_0} s^2 T_1^2\right] + t\left[\frac{\nu}{f_0} + \frac{\alpha}{f_0} sT_1\right] \end{aligned} \tag{123}$$

will be used for $D(t + sT_1)$ with an exactly analogous approximation for $D_a(t + sT_1)$. If these approximations are used in (121) and an appropriate change in variable is made, one finds that, for all practical purposes,

$$\begin{aligned} R_s(\tau, \nu, \alpha) = \{G(\xi, \eta)\}\xi &= \nu' + \alpha' sT_1 \\ \eta &= \tau' + \frac{\nu'}{f_0} sT_1 \\ &\quad + \frac{\alpha'}{2f_0} s^2 T_1^2, \end{aligned} \tag{124}$$

where $G(\xi, \eta)$ is given by (67). Since ξ, η are linear functions of τ', ν', α' one may quickly obtain the expression for $Q_s(\tau', \nu', \alpha')$ from a Taylor series expansion of $G(\xi, \eta)$ about $\xi = \eta = 0$ by appropriate substitution. The results of this substitution are listed below:

$$\begin{aligned} r_{11}^{(s)} &= G_{20}, \\ r_{22}^{(s)} &= G_{20}\left(\frac{s^2 T_1^2}{f_0^2}\right) + 2G_{11}\left(\frac{sT_1}{f_0}\right) + G_{02}, \\ r_{33}^{(s)} &= G_{20}\left(\frac{s^4 T_1^4}{4f_0^2}\right) + G_{11}\left(\frac{s^3 T_1^3}{f_0}\right) + G_{02}(s^2 T_1^2), \\ r_{12}^{(s)} &= G_{20}\left(\frac{sT_1}{f_0}\right) + G_{11}, \\ r_{13}^{(s)} &= G_{20}\left(\frac{s^2 T_1^2}{2f_0}\right) + G_{11} sT_1, \\ r_{23}^{(s)} &= G_{20}\left(\frac{s^3 T_1^3}{2f_0^2}\right) + G_{11}\left(\frac{3s^2 T_1^2}{2f_0}\right) + G_{02} sT_1. \end{aligned} \tag{125}$$

Use of (125) in (120) yields the expressions for r_{jk} given in (65).

Processing Gains Against Reverberation (Clutter) Using Matched Filters*

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Summary—The Woodward ambiguity function is discussed in connection with the output of a matched filter. A formula for the treatment of sonar reverberation or radar clutter is set up in terms of the ambiguity function. This formula is applied to determine the effect of signal waveform on the output signal-to-reverberation power ratio of a matched filter for a simple distribution of randomized scatterers.

Introduction

In radar and sonar, the ultimate goal in signal processing is to achieve a receiver output for which the desired signal echo stands out well above the accompanying noise and the clutter or reverberation[1] due to random scatterers in the transmitting medium. This goal provides a criterion for selecting the proper signal to transmit as well as the proper receiver for processing the returning echoes. When the background noise is Gaussian, the optimum receiver is a cross-correlator, or the equivalent matched filter, which correlates the input with a normalized replica of the expected signal echo. The expected echo-to-noise power ratio on the output of such a receiver is known to depend only on the total energy in the expected signal echo and the spectral power density of the expected noise [1]. Within the limitations imposed by noise considerations, variations in the waveform of the transmitted signal may be made to improve the output echo-to-reverberation power ratio.

A convenient mathematical tool for investigation of the echo-to-reverberation power ratio on the output of a matched filter is the signal ambiguity function used by Woodward [2] to study range and Doppler ambiguities for radar targets. Since the random scatterers producing reverberation may be considered as unwanted targets, this ambiguity function may be turned around to investigate the combined effect of scatterers at various ranges and velocities in obscuring the expected target at a particular range and velocity. The corresponding ambiguity diagrams [2], [4] have been used [1] to obtain qualitative graphic determinations of the effect of waveform in processing against reverberation. In the present paper, an analytic expression for the processing gain against reverberation is set up in terms of the signal ambiguity function, and the result is illustrated by a simple example. In this, as in the previous studies noted above, the ambiguity function used is based on the assumption that the carrier frequency for the transmitted signal is large compared to the signal bandwidth, so that for a given target velocity the Doppler frequency shift may be assumed to be essentially constant across the frequency band. It is planned to treat the more general case of broad-band signals in a later paper.

Background Theory: Ambiguity Functions

Suppose the transmitted signal is represented by the complex waveform [3]

$$Au(t) \exp (2\pi i f_0 t) \tag{1}$$

where $u(t)$ is normalized so that[2]

$$\int u(t)u^*(t)\, dt = 1. \tag{2}$$

Since the total energy $\mathcal{E}$ of this signal can be written[3]

$$\mathcal{E} = \tfrac{1}{2}A^2 \int |\, u(t)\,|^2\, dt,$$

it is seen that

$$A = (2\mathcal{E})^{1/2}. \tag{3}$$

Following Woodward,[4] the combined time and frequency autocorrelation function of $u(t)$ may be written in the complex forms

$$\chi(\tau, \phi) = \int u(t)u^*(t + \tau) \exp (-2\pi i \phi t)\, dt \tag{4a}$$

$$= \int U(f + \phi)U^*(f) \exp (-2\pi i f \tau)\, df \tag{4b}$$

where $U(f)$ is the complex spectrum of $u(t)$ defined by the Fourier Transform relation

$$U(f) = \mathfrak{F}_{tf} u(t) = \int u(t) \exp (-2\pi i f t)\, dt. \tag{5}$$

The corresponding signal ambiguity function may be written

$$\psi(\tau, \phi) = |\, \chi(\tau, \phi)\,|^2 = \chi(\tau, \phi)\chi^*(\tau, \phi). \tag{6}$$

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[1] In the present paper, the terms "clutter" and "reverberation" are used inter-changeably to refer to the combined echoes from large numbers of small random scatterers. The term "clutter" is more common to radar, and "reverberation" is the more common term in sonar.

[2] In this paper the superscript (*) is used to designate the complex conjugate, and all integrations are assumed to be taken over the significant region of the integrand unless otherwise indicated.
[3] Woodward [2], pp. 40, 41.
[4] Woodward [2], pp. 115–120.

t this point, some useful properties of the ambiguity unction will be derived.

From (4b) and (5), it is seen that

$$\chi(\tau, \phi) = \mathfrak{F}_{f\tau}[U(f + \phi)U^*(f)].$$

t follows that

$$\mathfrak{F}_{\tau\mu}\chi(\tau, \phi) = U(\phi - \mu)U^*(-\mu) \tag{7a}$$

nd

$$\mathfrak{F}_{\tau\mu}\chi^*(\tau, \phi) = U^*(\phi + \mu)U(\mu). \tag{7b}$$

Ience, the Fourier transform of the ambiguity function s given by the convolution integral

$$\mathfrak{F}_{\tau\mu}\psi(\tau, \phi)$$

$$= \int U(\phi - \nu)U^*(-\nu)U^*(\phi + \mu - \nu)U(\mu - \nu)\, d\nu. \tag{8}$$

ince ϕ and μ enter symmetrically in this last integral, follows that

$$\mathfrak{F}_{\tau\mu}\psi(\tau, \phi) = F_{\tau\phi}\psi(\tau, \mu) \tag{9}$$

r

$$\int \psi(\tau, \phi) \exp(-2\pi i\mu\tau)\, d\tau$$

$$= \int \psi(\tau, \mu) \exp(-2\pi i\phi\tau)\, d\tau, \tag{10}$$

hich could also have been derived from the essentially quivalent relation

$$\psi(\rho, \mu) = \mathfrak{F}^{-1}_{\phi\rho}\mathfrak{F}_{\tau\mu}\psi(\tau, \phi). \tag{11}$$

ue to Siebert [5], [6]. Taking $\mu = 0$ in (10) gives

$$\int \psi(\tau, \phi)\, d\tau = \int \psi(\tau, 0) \exp(-2\pi i\phi\tau)\, d\tau, \tag{12a}$$

relation which is found useful in a later section of the resent paper. Similarly, (4a) may be employed to obtain he relation

$$\int \psi(\tau, \mu)\, d\mu = \int \psi(0, \mu) \exp(2\pi i\mu\tau)\, d\mu. \tag{12b}$$

Taking ρ and μ each equal to zero in (11), we obtain

$$\mathfrak{F}^{-1}_{\phi 0}\mathfrak{F}_{\tau 0}\psi(\tau, \phi) = \psi(0, 0).$$

ince $\psi(0, 0)$ is unity from (2) and (4a), this relation can e stated in the integral form

$$\iint \psi(\tau, \phi)\, d\tau\, d\phi = 1. \tag{13}$$

q. (13) holds for any waveform $u(t)$ which satisfies (2), nd has been presented [2], [3] as a form of "uncertainty rinciple."

For the frequency-shifted and time-delayed waveform

$$v(t) = u(t - \tau_e) \exp[-2\pi i\phi_e(t - \tau_e)], \tag{14}$$

having the complex spectrum

$$V(f) = U(f + \phi_e) \exp(-2\pi i f\tau_e), \tag{15}$$

it is seen that the combined time and frequency autocorrelation function is given by

$$\chi_v(\tau, \phi) \equiv \int v(t)v^*(t + \tau) \exp(-2\pi i\phi t)\, dt$$

$$= \exp[2\pi i(\phi_e\tau - \phi\tau_e)]\chi(\tau, \phi), \tag{16}$$

where $\chi(\tau, \phi)$ is defined for $u(t)$ in (4).

Thus it is seen that

$$\begin{aligned}\psi_v(\tau, \phi) &\equiv \chi_v(\tau, \phi)\chi_v^*(\tau, \phi)\\ &= \chi(\tau, \phi)\chi^*(\tau, \phi)\\ &= \psi(\tau, \phi)\end{aligned} \tag{17}$$

where the symbols χ and ψ without subscripts apply to $u(t)$.

Treatment of Single Echoes

Consider a Doppler filter F_e which is matched for the signal waveform $v(t)$ defined in (14). Following Goldman,[5] the complex transfer function of F_e may be written

$$Y(f) = k_0 V^*(f) \exp(-2\pi i f t_0) \tag{18}$$

where k_0 is a constant to be evaluated later and t_0 is a time delay characteristic of the filter.

After being heterodyned down by the carrier frequency f_0, a received signal echo with time delay (two-way travel time) $\tau_s = \tau_e + \tau$ and total Doppler frequency shift $\phi_s = \phi_e + \phi$ may be represented by the waveform

$$w_s(t) = A_s v_s(t) \exp(i\beta_s) \tag{19}$$

where

$$v_s(t) = v(t - \tau) \exp[-2\pi i\phi(t - \tau)] \tag{20}$$

and where, corresponding to (3), the total energy in the signal echo is

$$\mathcal{E}_s = \frac{A_s^2}{2}. \tag{21}$$

The phase β_s is the sum of a number of phase terms, but the dominant term is $-2\pi f_0\tau_s$. In the present treatment, f_0 is assumed large compared to ϕ_s or ϕ_e, and τ_s is assumed to be of the same order of magnitude as τ_e. Thus, small and otherwise negligible adjustments in the value of τ_s may be made to compensate for the other phase terms in β_s. For simplicity, then, β_s may be replaced by $-2\pi f_0\tau_s$ if desired.

For the waveform $w_s(t)$ on the filter input, the corresponding output spectrum is

$$\begin{aligned}Z_s(f) &= W_s(f)Y(f)\\ &= k_0 A_s \exp(i\beta_s)V_s(f)V^*(f)\exp(-2\pi i f t_0)\\ &= k_0 A_s \exp(i\beta_s)V(f + \phi)V^*(f)\exp[-2\pi i f(\tau + t_0)]\end{aligned} \tag{22}$$

[5] Goldman [8], p. 232.

and the output waveform is

$$\begin{aligned} z_s(t) &= \mathfrak{F}_{ft}^{-1} Z_s(f) \\ &= k_0 A_s \exp(i\beta_s) \\ &\quad \cdot \int V(f+\phi) V^*(f) \exp[-2\pi i f(\tau + t_0 - t)]\, df \\ &= k_0 A_s \exp(i\beta_s) \chi_v(\tau + t_0 - t, \phi). \end{aligned} \tag{23}$$

The input power is represented by

$$S_i(t) = |w_s(t)|^2 = 2\mathcal{E}_s |v_s(t)|^2 \tag{24}$$

and the corresponding output power is represented by

$$S_0(t) = |z_s(t)|^2 = 2k_0^2 \mathcal{E}_s \psi(\tau + t_0 - t, \phi). \tag{25}$$

The total output energy is thus

$$\begin{aligned} \mathcal{E}_{0s} &= \tfrac{1}{2} \int S_0(t)\, dt \\ &= k_0^2 \mathcal{E}_s \int \psi(\tau + t_0 - t, \phi)\, dt \\ &= k_0^2 \mathcal{E}_s \int \psi(t, \phi)\, dt. \end{aligned} \tag{26}$$

If it is assumed that there is no loss in energy in the filter for a perfectly matched echo, it follows that

$$\begin{aligned} k_0^{-2} &= \int \psi(t, 0)\, dt \\ &= T_0 = \frac{1}{W} \end{aligned} \tag{27}$$

where T_0 is the time resolution constant and W the frequency span of the signal as defined by Woodward.[6] Replacing k_0^2 by W in (25) and (26) gives

$$S_0(t) = 2W\mathcal{E}_s \psi(\tau + t_0 - t, \phi) \tag{28}$$

and

$$\mathcal{E}_{0s} = W\mathcal{E}_s \int \psi(t, \phi)\, dt. \tag{29}$$

For $t = t_0$, (23) becomes

$$\begin{aligned} z_s(t_0) &= k_0 A_s \chi_v(\tau, \phi) \exp(i\beta_s) \\ &= k_0 A_s \chi(\tau, \phi) \exp(i\gamma_s) \end{aligned} \tag{30}$$

where, from (16),

$$\gamma_s = \beta_s + 2\pi(\phi_e \tau - \phi \tau_e). \tag{31}$$

As noted above, the dominant term in β_s is $-2\pi f_0 \tau_s$. For f_0 large compared to ϕ_e and ϕ_s, and τ_s the same order of magnitude as τ_e, it is seen that either β_s or γ_s (but of course not both) may be replaced by $-2\pi f_0 \tau_s$. Eq. (30) may thus be written as

$$z_s(t_0) = k_0 A_s \alpha_s \chi_s(\tau, \phi) \tag{32}$$

where

$$\alpha_s = \exp(-2\pi i f_0 \tau_s), \tag{33}$$

$$\tau \equiv \tau_s - \tau_e, \tag{34}$$

$$\phi \equiv \phi_s - \phi_e, \tag{35}$$

and where $\chi_s(\tau, \phi)$ may represent the combined time and frequency autocorrelation of either $u(t)$ or $v(t)$, but not of both.

The output power for F_e at the time $t = t_0$ ($t_0 + \tau_e$ after the emission of the signal) is thus represented by

$$S_0 = S_0(t_0) = 2W\mathcal{E}_s \psi(\tau, \phi). \tag{36}$$

This value of the output power may be physically achieved, at least in principle, by operating two identical F_e filters in quadrature and combining their squared outputs.

Before concluding this section, we wish to point out that the time average input power in an interval about $t = 0$ (time τ_e after emission of the signal) is more meaningful physically than the instantaneous input power as given by (24). This quantity can be written

$$\begin{aligned} \overline{S_i} &= \int S_i(t) |v(t)|^2\, dt \\ &= 2\mathcal{E}_s \int |v^*(t) v(t - \tau)|^2\, dt. \end{aligned} \tag{37}$$

A development similar to that used to obtain (8) yields

$$\int |u(t) u^*(t + \tau)|^2\, dt = \int \psi(\tau, \mu)\, d\mu \tag{38}$$

so that

$$\overline{S_i} = 2\mathcal{E}_s \int \psi(\tau, \mu)\, d\mu. \tag{39}$$

Treatment of Multiple Echoes

Suppose now that instead of a single echo, the received signal consists of the sum of N overlapping echoes arising from extended targets, multiple targets or multipath conditions. Replacing the index s by n in (19) and letting n take on the integral values from 1 to N, the waveform on the input to the filter F_e may be represented by

$$w(t) = \sum_n w_n(t) = \sum_n A_n v_n(t) \exp(i\beta_n) \tag{40}$$

and the input power by

$$\begin{aligned} |w(t)|^2 &= \sum_n \sum_m w_n(t) w_m^*(t) \\ &= \overline{R_i}(t) + \tilde{R}_i(t) \end{aligned} \tag{41}$$

[6] Woodward [2], pp. 117–119.

where

$$v_n(t) \equiv v(t - \tau_{ne}) \exp[-2\pi i\phi_{ne}(t - \tau_{ne})], \tag{42}$$

$$\tau_{ne} \equiv \tau_n - \tau_e , \tag{43}$$

$$\phi_{ne} \equiv \phi_n - \phi_e , \tag{44}$$

$$\overline{R_i}(t) \equiv \sum_n |w_n(t)|^2 = \sum_n A_n^2 |v_n(t)|^2$$

$$= 2 \sum_n \mathcal{E}_n |v(t - \tau_{ne})|^2, \tag{45}$$

and

$$\tilde{R}_i(t) \equiv \sum_n \sum_{m \neq n} w_n(t) w_m^*(t)$$

$$= \sum_n \sum_{m \neq n} A_n A_m v_n(t) v_m^*(t) \exp[i(\beta_n - \beta_m)]. \tag{46}$$

Corresponding to (32), the filter output at $t = t_0$ (a time interval $t_0 + \tau_e$ after the emission of the signal) is represented by

$$z(t_0) = \sum_n z_n(t_0)$$

$$= k_0 \sum_n A_n \alpha_n \chi_n \tag{47}$$

where

$$\alpha_n \equiv \exp(-2\pi i f_0 \tau_n) \tag{48}$$

and

$$\chi_n \equiv \chi(\tau_{ne}, \phi_{ne}). \tag{49}$$

The corresponding output power is then represented by

$$|z(t_0)|^2 = \sum_n \sum_m z_n(t_0) z_m^*(t_0)$$

$$= \overline{R_0}(t_0) + \tilde{R}_0(t_0), \tag{50}$$

where

$$\overline{R_0}(t_0) \equiv \sum_n |z_n(t_0)|^2 = 2W \sum_n \mathcal{E}_n \psi_n , \tag{51}$$

$$\tilde{R}_0(t_0) \equiv \sum_n \sum_{m \neq n} z_n(t_0) z_m^*(t_0)$$

$$= W \sum_n \sum_{m \neq n} A_n A_m \alpha_n \alpha_m^* \chi_n \chi_m^* , \tag{52}$$

and

$$\psi_n \equiv \psi(\tau_{ne}, \phi_{ne}). \tag{53}$$

Treatment of Reverberation (Clutter)

In the case of reverberation, where the received signal at any particular time is assumed to consist of a large number of small overlapping unresolvable echoes, it is impractical to consider each individual echo as in (40) and (47). Instead, it becomes necessary to deal with statistical averages (expected values).

For this purpose, consider $\overline{R_i}(t)$ as given by (45). The individual echoes with energies $\mathcal{E}_n$ are now assumed to be coming from large numbers of individual scatterers randomly distributed in range, velocity, and acceleration. For the purposes of the present paper, it is assumed that the range and velocity (or the equivalent time delay and Doppler frequency shift) for a particular scatterer are essentially constant over the effective duration of the signal. For a given physical situation, this may place an upper limit on the signal durations for which the present discussion is applicable. In general, however, if the signal is repeated at intervals which are long compared to the signal duration, the values of time delay and frequency shift for a particular scatterer may be expected to change radically during the interval between signals. In order to obtain a representative expression for the input power, it is therefore convenient to replace the discrete energy contributions $\mathcal{E}_n$ from the individual scatterers by a continuous function of time delay and frequency shift. For a given small area $d\tau\, d\phi$ at $(\tau_e + \tau, \phi_e + \phi)$ in the (τ, ϕ) − plane, it is thus seen that the returned energy may be expected to vary radically from one signal to the next. Representing the expected value (mean) of the reverberation energy from this area as $\mathcal{E}(\tau_e + \tau, \phi_e + \phi)\, d\tau\, d\phi$, where $\mathcal{E}(\tau_e + \tau, \phi_e + \phi)$ is assumed to be a continuous variable in τ and ϕ, it is seen that (45) may be replaced by

$$\overline{R_i}(t) = 2 \iint \mathcal{E}(\tau_e + \tau, \phi_e + \phi) |v(t - \tau)|^2 \, d\tau\, d\phi$$

$$= 2 \int \mathcal{E}(\tau_e + \tau) |v(t - \tau)|^2 \, d\tau \tag{54}$$

where now $\mathcal{E}(\tau_e + \tau)\, d\tau$ is the mean reverberation energy returned to the receiver from a strip of width $d\tau$ in the (τ, ϕ)-plane extending from $\tau_e + \tau$ to $\tau_e + \tau + d\tau$.

Because of the random phasing of the cross-product terms in (46), $\tilde{R}_i(t)$ is a randomly fluctuating term with zero mean. The rms value of $\tilde{R}_i(t)$ is of course not zero, but it is not being considered in the present paper.

Corresponding to (39), the time average of the mean input reverberation power in an interval about $t = 0$ (time τ_e after emission of the signal) is

$$\overline{R_i} = \int \overline{R_i}(t) |v(t)|^2 \, dt$$

$$= 2 \iint \mathcal{E}(\tau_e + \tau) \psi(\tau, \mu) \, d\tau\, d\mu. \tag{55}$$

The mean output power in (51) may be written in the integral form as

$$\overline{R_i} \equiv \overline{R_0}(t_0)$$

$$= 2W \iint \mathcal{E}(\tau_e + \tau, \phi_e + \phi) \psi(\tau, \phi) \, d\tau\, d\phi. \tag{56}$$

As with $\tilde{R}_i(t)$, the mean value of $\tilde{R}_0(t_0)$ in (52) is zero. Although the rms value is not zero, it is not being treated in the present paper, and random fluctuations of the output power about the mean value are ignored.

Case of Separable Distributions

In many, if not most, cases of practical interest, it is possible to take

$$\mathcal{E}(\tau_e + \tau, \phi_e + \phi) = \mathcal{E}(\tau_e + \tau)Q(\phi_e + \phi) \tag{57}$$

where

$$\int Q(\phi_e + \phi)\, d\phi = \int Q(\phi)\, d\phi = 1. \tag{58}$$

It should be noted that even though $Q(\phi)$ is assumed to be essentially independent of τ in the region of interest about τ_e, this function may be different for different values of τ_e. $\mathcal{E}(\tau_e + \tau)\, d\tau$ has the same interpretation as in (54).

For the above assumptions, (55) remains unchanged

$$\overline{R_i} = 2 \iint \mathcal{E}(\tau_e + \tau)\psi(\tau, \mu)\, d\tau\, d\mu, \tag{59}$$

but (56) becomes

$$\overline{R_0} = 2W \iint \mathcal{E}(\tau_e + \tau)Q(\phi_e + \phi)\psi(\tau, \phi)\, d\tau\, d\phi. \tag{60}$$

In the usual case, $\mathcal{E}(\tau_e + \tau)$ may also be taken as essentially constant over the region of interest about τ_e. Taking this constant value as

$$\mathcal{E}_e \equiv \mathcal{E}(\tau_e) = \mathcal{E}(\tau_e + \tau) \tag{61}$$

and making use of (2), it is seen that (59) reduces to

$$\overline{R_i} = 2\mathcal{E}_e \iint \psi(\tau, \mu)\, d\tau\, d\mu = 2\mathcal{E}_e \tag{62}$$

and (60) becomes

$$\overline{R_0} = 2W\mathcal{E}_e \iint Q(\phi_e + \phi)\psi(\tau, \phi)\, d\tau\, d\phi. \tag{63}$$

From (12) and (62), this becomes

$$\begin{aligned}\overline{R_0} &= \overline{R_i}W \int Q(\phi_e + \phi) \int \psi(\tau, 0) \exp(-2\pi i\phi\tau)\, d\tau\, d\phi \\ &= \overline{R_i}W \int \psi(\tau, 0) \int Q(\phi_e + \phi) \exp(-2\pi i\phi\tau)\, d\phi\, d\tau \\ &= \overline{R_i}W \int \psi(\tau, 0)q(-\tau) \exp(2\pi i\phi_e\tau)\, d\tau \end{aligned} \tag{64}$$

where

$$q(\tau) = \mathfrak{F}_{\phi\tau}^{-1}Q(\phi). \tag{65}$$

Since

$$\psi(\tau, 0) = \psi(-\tau, 0) \tag{66}$$

it is seen that (64) can be written

$$\begin{aligned}\overline{R_0} &= \overline{R_i}W \int \psi(\tau, 0)q(\tau) \exp(-2\pi i\phi_e\tau)\, d\tau \\ &= \overline{R_i}W\mathfrak{F}_{\tau\phi_e}g(\tau) = \overline{R_i}WG(\phi_e)\end{aligned} \tag{67}$$

where

$$g(\tau) = \psi(\tau, 0)q(\tau). \tag{68}$$

Signal-to-Reverberation Power Ratios

For a signal echo that is perfectly matched by the filter F_e, it is seen from (37) and (39) that the time average of the input power is

$$\begin{aligned}\overline{S_i} &= 2\mathcal{E}_s \int |v(t)|^4\, dt \\ &= 2\mathcal{E}_s \int \psi(0, \mu)\, d\mu \\ &= 2\mathcal{E}_s/T\end{aligned} \tag{69}$$

where, following Woodward,[6] T is the nominal time span of the signal. The corresponding output power, from (36), may be written

$$\begin{aligned}S_0 &= 2W\mathcal{E}_s\psi(0, 0) \\ &= 2W\mathcal{E}_s .\end{aligned} \tag{70}$$

Suppose the processing gain against reverberation is represented by

$$\begin{aligned}K_e &= (S_0/\overline{R_0})/(\overline{S_i}/\overline{R_i}) \\ &= S_0\overline{R_i}/\overline{S_i}\overline{R_0} .\end{aligned} \tag{71}$$

From (55), (56), (69) and (70), this becomes

$$K_e = \frac{T \iint \mathcal{E}(\tau_e + \tau)\psi(\tau, \mu)\, d\tau\, d\mu}{\iint \mathcal{E}(\tau_e + \tau, \phi_e + \phi)\psi(\tau, \phi)\, d\tau\, d\phi}. \tag{72}$$

For the special case where the energy distribution of the reverberation is separable, (59) and (60) apply, and (72) becomes

$$K_e = \frac{T \iint \mathcal{E}(\tau_e + \tau)\psi(\tau, \mu)\, d\tau\, d\mu}{\iint \mathcal{E}(\tau_e + \tau)Q(\phi_e + \phi)\psi(\tau, \phi)\, d\tau\, d\phi}. \tag{73}$$

And for the case of constant energy distribution with respect to τ, (62) and (67) apply, and the processing gain against reverberation becomes

$$K_e = T/G(\phi_e). \tag{74}$$

Usually one is interested in the output signal-to-reverberation power ratio for a given situation. From (71), this may be written

$$S_0/\overline{R_0} = K_e(\overline{S_i}/\overline{R_i}). \tag{75}$$

For the particular case of perfectly matched signal echo and constant reverberation energy distribution with respect to τ, this becomes

$$S_0/\overline{R_0} = K_e(\mathcal{E}_s/T\mathcal{E}_e). \tag{76}$$

Illustrative Example: Gaussian-Shaped Power Spectrum

For a signal with a Gaussian-shaped power spectrum, ne squared magnitude of the time correlation function Gaussian-shaped also:

$$\psi(\tau, 0) = \exp(-\pi W^2 \tau^2), \tag{77}$$

here W is the nominal frequency span of the signal as ı (27).

Assume the distribution of reverberation energy with espect to velocity to be Gaussian also, so that

$$Q(\phi) = (1/W_q) \exp(-\pi\phi^2/W_q^2) \tag{78}$$

here W_q is the nominal spread of Doppler frequency ıifts in the reverberation. Then, from (65),

$$q(\tau) = \exp(-\pi\tau^2 W_q^2) \tag{79}$$

ıd, from (68),

$$g(\tau) = \exp(-\pi W_g^2 \tau^2) \tag{80}$$

here

$$W_g^2 = W^2 + W_q^2 = W_q^2(1 + W^2/W_q^2). \tag{81}$$

his gives

$$G(\phi_e) = (1/W_g) \exp(-\pi\phi_e^2/W_g^2). \tag{82}$$

rom (74), the processing gain against reverberation for ıis particular case becomes

$$K_e = TW_g \exp(\pi\phi_e^2/W_g^2). \tag{83}$$

For convenience in plotting, both sides of (83) are ıvided by TW_q and the result is written in the form

$$K_e/TW_q = (W_g/W_q) \exp[\pi(\phi_e/W_q)^2/(W_g/W_q)^2]. \tag{84}$$

rom (81), it is seen that

$$W_g/W_q = (1 + W^2/W_q^2)^{1/2}. \tag{85}$$

hus K_e/TW_q may be plotted as a function of W/W_q r various values of ϕ_e/W_q. This has been done in Fig. 1 ter conversion to db. Taking μ_0 to represent the adjusted ı value, we have

$$\begin{aligned} \mu_0 &= 10 \log (K_e/TW_q) \\ &= -10 \log [W_q G(\phi_e)]. \end{aligned} \tag{86}$$

om (76), it is seen that

$$\begin{aligned} \mu_0 &= 10 \log [(S_0/\overline{R_0})(\mathcal{E}_e/\mathcal{E}_s W_q)] \\ &= 10 \log (S_0/\overline{R_0}) + 10 \log (\mathcal{E}_e/\mathcal{E}_s W_q). \end{aligned} \tag{87}$$

is noted that (87) is dimensionally correct, since $\mathcal{E}_e$ tually has the dimensions of energy per unit time delay. ssuming $\mathcal{E}_e$, $\mathcal{E}_s$ and W_q to be known for a particular ıysical situation, it is seen from (87) that the output gnal-to-reverberation power ratio for given values of and W may be determined from Fig. 1. The constant $_e/\mathcal{E}_s W_q$) affects the zero db reference point but not the ape of the curves.

Interpretation of Fig. 1

The general trends of the curves in Fig. 1 will be considered in relation to the relatively simple situation illustrated in Fig. 2. Here a distribution $\mathcal{E}(\tau_s, \phi_s)$ of reverberation energy and the elliptical ambiguity function $\psi(\tau, \phi)$ of a short "single-frequency" pulse are represented by their "3 db down" contours in the (τ_s, ϕ_s)-plane. (See Stewart and Westerfield [1] for more complete details on this type of representation in terms of ambiguity diagrams.)

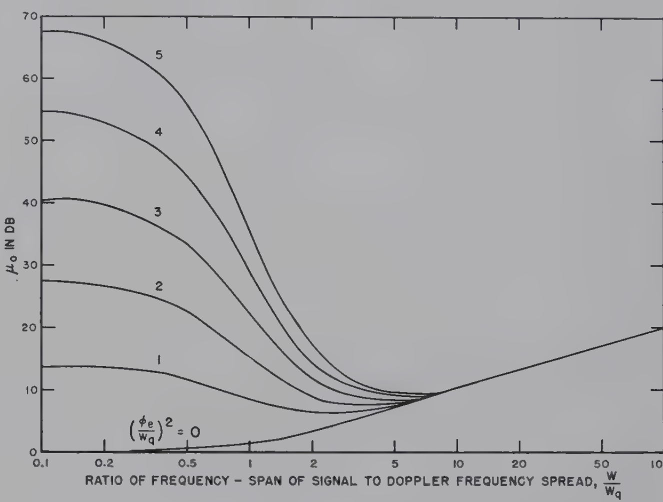

Fig. 1—μ_0 plotted as a function of signal bandwidth for various Doppler frequency shifts.

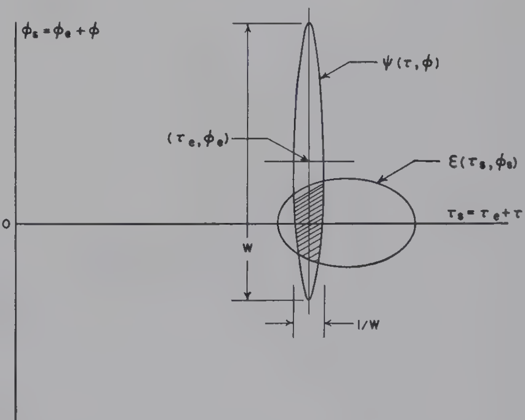

Fig. 2—Reverberation energy distribution and signal ambiguity function illustrated by their nominal extents in the (τ_s, ϕ_s)—plane.

The upward trend of μ_0 for large values of signal bandwidth (*i.e.*, large values of W/W_q) is due to the improved range resolution associated with the decrease in time resolution $T_0 = 1/W$ which represents the extent of the ambiguity function along the τ_s (delay time) axis. To state it another way, the intersection of the ambiguity function and the reverberation energy distribution is small for large W. This intersection is given by the volume integral in the denominator of (72). From a detection

stand-point, this effect is most important when attempting to detect a "target" having small Doppler frequency shift. In fact, for ϕ_e in the vicinity of zero, one has no alternative but to increase W in order to increase μ_0. Urkowitz [9] has treated the optimum filter for this case.

In a similar fashion, the upward trend of μ_0 for small values of signal bandwidth is associated with a flattening of the ellipse so as to have small extent along the ϕ_s axis. In this case, the effect is clearly most pronounced for targets having large Doppler frequency shifts. The transmitted signal is a narrow-band pulse which is of long duration since $T \geq 1/W$.

The most striking feature in Fig. 1 is the presence of pronounced minima in all curves except those for very small Doppler shifts. The minimum of a particular curve occurs at that bandwidth for which the intersection of the ambiguity function and the reverberation energy distribution is largest.

It is noted that the output signal-to-reverberation power ratio, represented by μ_0 in Fig. 1, does not depend on T.

Conclusion

The striking feature of the curves in Fig. 1 is the pronounced valley exhibited by all curves for nonzero values of ϕ_e (Doppler frequency shift of signal echo). From the standpoint of obtaining large values of μ_0 (output signal-to-reverberation power ratio in db for matched filter), it is necessary to employ either large or small values of W/W_q (ratio of transmitted signal bandwidth to spread of Doppler frequency shifts in reverberation), depending on the value of ϕ_e/W_q. In no case is the mid-region of W/W_q best in this regard.

If the scattering distribution is different from Gaussian and/or if the power spectrum of the transmitted signal is different from Gaussian in shape, graphs corresponding to Fig. 1 may be plotted to suit these new conditions. The Gaussian power spectrum for the signal and the Gaussian velocity distribution for the scatterers were chosen in this particular example because of the resulting simplicity in the computations.

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On New Classes of Matched Filters and Generalizations of the Matched Filter Concept*

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Summary—**In this paper it is shown how the earlier concepts of the matched filter may be generalized by recognizing explicitly the decision-making character of most reception systems. Accordingly, when an approach making use of statistical decision theory is applied for both signal detection and extraction, a variety of new classes of matched filters (Bayes matched filters) can be defined. These can be described specifically in the critical situation of threshold reception, where system optimality is at a premium. It is shown, for incoherent reception in some important special instances, that matched filters based on maximizing output signal-to-noise ratio (the S/N matched filters of the earlier theory) are also optimum from the broader, decision viewpoint. The required optimum filters are themselves time-varying and nonunique, and thus permit a measure of design freedom. In all instances, realizable filters are possible, and it is shown how their weighting functions may be determined. Both discrete and continuous filtering on a finite interval, (O, T), are considered.**

I. Introduction

THE general concept of a matched filter[1] in which "matching" is a form of optimization which attempts to enhance in some appropriate sense the reception of desired signals in undesired noise backgrounds is now well known. Reception itself may consist of a detection process, where "yes" or "no" as to presence or absence of the desired signal is the desired system output; or reception may be an extraction, or estimation operation, where the intent is to measure some signal waveform or information-bearing parameter of the signal. Matched filters themselves, in this study as well as in earlier definitions, are required to be linear, though not necessarily invariant or realizable (in the usual technical sense of operating only on the "past" of the input disturbance), and are used in conjunction with (possibly) subsequent zero-memory nonlinear elements and linear "postdetection" integrating filters. However, it should be emphasized at the outset that, unlike earlier definitions, matched filtering in the present definition implies a more general relationship between filter structure, noise, and signal characteristics, and includes the earlier class as special cases, as we shall see presently. The structure of matched filters in general depends on, 1) the character of the signal process in reception; 2) the statistical properties of the accompanying noise (and the manner in which it combines with the signal process); and 3) in particular, the criterion of optimality which is selected. In most instances 1) and 2) are *a priori* data; accordingly, by modifying and broadening the third element, *i.e.*, the criterion itself, we can expect to generate new and wider classes of matched filters. How this may be achieved and, specifically, what new types of such filters can be defined and described is the main purpose of the present paper.

The earlier classes of matched filters have all been defined essentially on an energy basis, *i.e.*, on some form of maximization of signal energy *vis-à-vis* that of the accompanying noise, without direct reference to the actual decision process implied in reception. Usually, these matched filters have been derived by maximizing a signal-to-noise (power or intensity) ratio S/N, where one seeks to maximize a peak value (at some specified point in time) of an output signal against a given (rms) noise level (see the original work of North,[2] Van Vleck and Middleton,[1] and others.[3-11]

Strictly speaking, such a formulation really belongs to reception as an estimation process,[12] but since maximizing S/N also bears a monotonic relation to a natural model of the detection mechanism, it has been reasonably applied to detection, as well. The essential new feature of the present approach is the recognition of reception as a *decision* process (*i.e.*, a "yes"-"no" or measurement operation), and the consequent use of statistical decision theory to provide criteria of optimality upon which, in turn, a broad generalization of the matched-filter concept can be founded.[13] Here the principal results of this broadening

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[13] *Ibid.*, sec, 20.2–4; also Chaps. 18, 19, and 21 (for signal extraction).

of the criterion is the creation of a number of new classes of matched filters,[13] whose structure, appropriate to the reception situation (*e.g.*, detection or extraction), can be described explicitly in the critical condition of threshold, or weak-signal operation.[14] Although optimum *threshold* operation essentially yields an energy (*i.e.*, quadratic) form for receiver structure, and hence linear filters may once more be expected as optimizing elements, the precise nature of these Bayes matched filters (as we shall call them from the criterion from which they are derived) is often quite different from the earlier class of S/N-matched filters, which are also based on an energy (*i.e.*, quadratic) form. This is not surprising, since the criteria of optimality in the two cases are so different. In what follows, Section II provides a short introduction to optimal threshold reception; Section III contains the main material of the paper, the definitions and derivations of several classes of Bayes matched filters; and Section IV briefly reviews the differences and occasional similarities of these Bayes and the earlier S/N matched filters.

II. Optimum (Threshold) Reception[15]

A few remarks on reception as a decision process are needed to indicate the general form of optimum system structure, from which, in turn, these Bayes matched filters are obtained. First, in a decision system certain costs or numerical value judgments are assigned to the various possible correct and incorrect outcomes. Optimum or Bayes reception is then defined as that operation on received data which minimizes the average cost of decision; receivers which do this are called Bayes receivers.[16,17] Next, let us consider detection. Without specifying particular statistics, we make the somewhat unrestrictive assumptions that the input signal and noise processes are additive and independent.[18–21] In order to discuss specific structures, let us make the further assumption of threshold operation, where the level of the input signal (when on) is comparable to or less than that of the background noise. This is almost always satisfactory for design and evaluation purposes, since systems optimized for weak signal performance surpass this on an absolute basis at high signal levels (although they are no longer optimum for such levels), where strict optimality is rarely required anyway, provided, of course, that care is taken to avoid saturation, or similar effects, which might suppress the information-bearing features of the signal.

For detection of the presence or absence of a signal in noise, it is well known that the optimum structure is embodied in the (logarithm of the) generalized likelihood ratio,[11,17] namely,

$$\log \Lambda_n(\mathbf{v}) = \log \{p\langle F_n(\mathbf{V} \mid \mathbf{S})\rangle / qF_n(\mathbf{V} \mid \mathbf{0})\} \tag{1}$$

for discrete data sampling, where $\mathbf{V} = (V_1, V_2, \cdots, V_n)$, $V_k = V(t_k)$, $t_1 \leq t_2 \leq \cdots < t_n$, represent the received data, now expressed as a vector, on the data acquisition or observation period $(0, T)$. Similarly, the signal is $\mathbf{S} = (S_1, S_2, \cdots, S_n)$; and $\langle\ \rangle$ in (1) denotes the statistical average over $\mathbf{S}$, or as is more often the case, over the random parameters, $\boldsymbol{\theta}$, of $\mathbf{S} = \mathbf{S}(\boldsymbol{\theta})$, when the waveform is known. In the usual way, p, $q(=1-p)$ are the *a priori* probabilities of the states $H_1 : V = S + N$ or $H_0 : V = N$, on any one sample $\mathbf{V}$, while F_n is the conditional distribution density of $\mathbf{V}$, given $\mathbf{S}$. When continuous data processing is employed in $(0, T)$, the likelihood ratio (1) is replaced by the likelihood ratio functional

$$\log \Lambda_T(V(t) = \lim_{n\to\infty} \log \Lambda_n(\mathbf{V}) \qquad (0 \leq t \leq T) \tag{2}$$

when this limit exists.[22]

The optimum structure in threshold operation is then obtained by a development of (1) in a power series of the input signal-to-noise ratio a_0, (or its mean, or rms value), defined by

$$a_0 = A_0/\sqrt{2\psi}; \qquad \psi = \overline{N^2}(\bar{N} = 0);$$

$$\frac{A_0^2}{2} = \frac{1}{T}\int_0^T S(t, \boldsymbol{\theta})^2\, dt, \tag{3a}$$

where ψ is the mean background noise intensity. Introducing the further normalizations

$$S(t) = \sqrt{\psi}\, a_0 S(t); \qquad V(t) = \sqrt{\psi}\, v(t), \tag{3b}$$

we find for discrete sampling on $(0, T)$ that the threshold expansion is of the type[23]

$$\log \Lambda_n = B_0 + B_1\tilde{\mathbf{v}}\mathbf{C}^{(1)}\langle\mathbf{s}\rangle + B_2\tilde{\mathbf{v}}\langle\mathbf{C}^{(2)}\rangle\mathbf{v} + 0(a_0^3, S^3; v, v^2, v^3) \tag{4}$$

when $\mathbf{C}^{(1)}$, $\langle\mathbf{C}^{(2)}\rangle$ are symmetric, positive definite $(n \times n)$ matrices. (The positive definiteness and symmetry reflect the physical observation that optimum detection systems are energy discriminating devices, at least.) Here $C^{(1)}$

[14] In special cases of interest, explicit results for general signal levels are also sometimes possible.

[15] Middleton, *op. cit.*, sec. 19.4 and Chap. 21.

[16] D. Middleton and D. Van Meter, "Detection and extraction of signals in noise from the point of view of statistical decision theory," *Soc. Indus. and Appl. Math.*, vol. 3, p. 192, December, 1955; vol. 4, p. 86, June, 1956.

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[18] In certain scatter multipath situations, and in the case of clutter, one has, instead, a multiplicated process for the noise, which now is strongly influenced by the original signal. However, it is usually possible to treat such interference as additive (and specifically Gaussian) after suitable modification of the primary noise mechanism.

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[22] *i.e.*, when the regular case occurs, as it does in all physical applications, for both deterministic and stochastic signals. See also Middleton, *op. cit.*, footnote 11, sec. 19.4–3 and references.

[23] Although this is in somewhat different form from that usually given, the derivation is unchanged from that of the original development of Middleton, *op. cit.*, footnote 11, (19.50) and (19.56).

$\gamma^{(2)}$, etc., are determined by the noise statistics and the manner in which signal and noise interact. Specific examples in the common situation of normal noise backgrounds are cited in (61) and (64a), subsequently. The dominating terms of B_0, B_1, B_2 are $0(a_0^0$, or a_0, or $a_0^2)$, $0(a_0)$, $0(a_0^2)$, respectively.[24] For continuous sampling on $(0, T)$, (4) becomes

$$\log \Lambda_T[V(t)] = B_{0T} + B_{1T} \iint_{-\infty}^{\infty} v(t) G_T^{(1)}(t, u) \langle s(u, \theta) \rangle \, dt \, du$$

$$+ B_{2T} \iint_{-\infty}^{\infty} v(t) \langle G^{(2)}(t, u) \rangle v(u) \, dt \, du \cdots + 0(a_0^3, a_0^4) \quad (5)$$

where B_{0T}, B_{1T}, B_{2T}, etc., $G_T^{(1)}$, $\langle G_T^{(2)} \rangle$ are the appropriate limiting forms[25] of $B_0, \cdots, \mathbf{C}^{(1)}$, etc., and $G_T^{(1)}, \langle G_T^{(2)} \rangle = 0 \cdots$, outside $(0-, T+)$.

The corresponding theory for optimum extraction is considerably more involved. For although the criterion of optimality is still the minimization of average risk, a much wider class of cost functions is now possible, depending on the particular problem. This is in effect an additional degree of freedom for the designer, and noticeably affects the explicit form of the resulting Bayes estimators. Thus, if $\gamma_\sigma(\mathbf{S} \mid \mathbf{V})$ is the estimator of $\mathbf{S}$ given $\mathbf{V}$, then one common choice of cost function is $C(\mathbf{S}, \gamma_\sigma) = C_0 \| \mathbf{S} - \gamma_\sigma(\mathbf{S} \mid \mathbf{V}) \|^2$, the familiar mean-square error estimate. The Bayes, or optimum estimator γ_σ^* here is the well-known conditional expectation of $\mathbf{S}$, given $\mathbf{V}$,[26] (footnote 11, Eq. 21.5, for example) and threshold operation yields a development of the type (4), or (5) (with, of course, different bias and weightings). Difference cost-functions of a degree other than quadratic can also be shown to give optimum threshold estimation of this same general form. However, when a "simple" cost function like

$$C(\mathbf{S}, \gamma_\sigma) = C_0 \left\{ A_m - \sum_{m=1}^{M} C_m \, \delta[S_m - (\gamma_\sigma)_m] \right\},$$

[cf. Eq. (21.56), Ref. 11,] is chosen, the Bayes estimator is also the maximum (unconditional) likelihood estimator, which does not directly lead to a threshold development like (4), or (5), unless the quantities being estimated (waveform, or amplitude) appear *linearly* in the signal. In spite of the more involved nature of optimum estimation procedures, we may say again that for threshold operation quadratic forms in v, v^2 are the essential operations on the received data, yielding numbers which are then subjected to various purely computational manipulations, such as multiplication, raising to powers, evaluation of Bessel functions, hypergometric functions, and the like, with these numbers as their arguments.[27]

Before we examine the basic quadratic forms that, as we have now seen, arise in an optimal threshold theory, let us make a further observation concerning reception, which usually simplifies the optimum structures [cf. (4) and (5)] somewhat. If we represent the (normalized) signal process by $s = s(t - \epsilon, \boldsymbol{\theta}')$, where ϵ is the *epoch*, or time, relating some representative point on the wave to the receiver's time-scale, we can distinguish three principal conditions of operation: 1) *Coherent* reception, with *sample* certainty, *i.e.*, signal epoch is known precisely, so that $\langle s \rangle_\epsilon \neq 0$, and consequently the term in B_1, or B_{1T}, which is $0(a_0)$, is the dominating one. 2) Pure, *incoherent observation*, with *sample uncertainty*, where ϵ is uniformly distributed over the duration of the signal if broad band, or at least over the duration of an RF cycle, if narrow band, so that $\langle s \rangle_\epsilon \doteq 0$, and the dominant term is now $0(a_0^2)$, cf. B_2, B_{2T} and their coefficients above. 3) Finally, there is partially coherent observation, where ϵ is not uniformly distributed or such that $\langle s \rangle_\epsilon$ vanishes; then both first- and second-order terms in the structure are required, and the optimum threshold system is consequently a mixture of linear and nonlinear operations on the received data. Thus, coherent reception involves a generalized, averaged cross-correlation of the received data with the known signal waveform, while incoherent reception requires a generalized, averaged autocorrelation of the received wave with itself.[28]

To summarize, for optimum (or Bayes) threshold reception of deterministic signals in general additive and independent noise, processing of the received data depends on either a generalized and averaged cross- or autocorrelation, embodied in the quadratic forms

$$\langle \Phi_n(v, s) \rangle = \tilde{\mathbf{v}} \mathbf{C}^{(1)} \langle \mathbf{s} \rangle; \qquad \langle \Psi_n \rangle = \tilde{\mathbf{v}} \langle \mathbf{C}^{(2)} \rangle \mathbf{v} \quad (6)$$

for discrete sampling, and in

$$\langle \Phi_T(v, s) \rangle = \iint_{-\infty}^{\infty} v(t) G_T^{(1)}(t, u) \langle s(u; \boldsymbol{\theta}) \rangle \, dt \, du, \quad (7a)$$

$$\langle \Psi_T \rangle = \iint_{-\infty}^{\infty} v(t) \langle G_T^{(2)}(t, u) \rangle v(u) \, dt \, du, \quad (7b)$$

for continuous sampling in $(0, T)$. As we shall see below, it is by resolving these expressions into sequences of

[24] Compare this with the development of Middleton, *op. cit.*, footnote 11, (19.50) and (19.56). We observe also that there are higher order terms in v $(0s; v^2, v^3, \cdots)$ for B_1, $0(s^2, v^3, v^4, \cdots)$ for B_2, etc. However, with the present threshold development, for terms in v of higher order in B_1 (*vis*: $v^2, v^3, \cdots$), which is otherwise $0(s, a_0, \cdots)$, we take the ensemble average over v in these terms with respect to the null hypotheses H_0, remembering that $v = a_0 s + n$, and collect the resulting terms $0(a_0^2, a_0^3, \cdots)$ in the "bias" B_0. Because of the weak signal assumption, this procedure yields only slight departures from optimality, which can be ignored in most applications. Note, too, that this is particularly true for incoherent reception; see the discussion of footnote 11, sec. 19.4.

[25] For a discussion of the limiting process, see Middleton, *op. cit.*, footnote 11, sec. 19.4–2.

[26] Middleton, *op. cit.*, footnote 17, (4.8).

[27] Middleton, *op. cit.*, footnote 11, see sec. 21.3–2,3 where a fairly complete theory of Bayes estimation of the amplitude of a deterministic signal in narrow-band normal noise for incoherent reception is developed, using a quadratic and a simple cost function (maximum likelihood estimation).

[28] For a more comprehensive statement, applied to detection only, see Middleton, *op. cit.*, footnote 11, sec. 19.4–3, and D. Middleton and D. Van Meter, *op. cit.*, sec. 3.7.

linear and [for (7b)] nonlinear operations, that new classes of matched filters, which we have called Bayes matched filters, can be defined. Moreover, from the preceding remarks it is clear that such filters possess optimal properties, since they arise in optimum decision systems for both detection and extraction, which minimizes the average cost of incorrect decisions.

III. Bayes Matched Filters for Threshold Reception

Our central task now is to resolve the quadratic forms (6) and (7) into ordered sequences of circuit operations which can be interpreted as combinations of optimum linear filters, nonlinear devices, integrating or averaging elements, and the like. For this purpose we shall start with discrete filters, corresponding to the discrete sampling procedures implied in (6), and pass to the appropriate limit ($n \to \infty$) on $(0, T)$ for the continuous cases.

For all types of Bayes matched filters we require generally that:

1) they be linear (but not necessarily invariant or realizable);
2) they precede other (possible) *zero-memory* nonlinear elements, and any subsequent (linear) integrations.

There will be several classes of these filters, depending on whether the observation (*i.e.*, data acquisition) process is coherent or not (cf. Sec. II above), and on that fact that quadratic forms like Φ_n, Ψ_n, etc., are resolvable into ordered sequences of elements in more than one way, even subject to the general requirements 1) and 2) above. Some are realizable,[29] some not; all to some extent involve time-varying components, since we are dealing (realistically) with finite data samples.

Let us consider first the simpler situation of coherent operation, so that $\langle\Phi_n\rangle$, $\langle\Phi_T\rangle$, [cf. (6) and (7a)] are the quantities of interest.

A. Coherent Operation, $\langle\Phi_n\rangle$, $\langle\Phi_T\rangle$

We begin with the quadratic form[30] $\tilde{\mathbf{v}}\mathbf{C}^{(1)}\bar{\mathbf{s}}$, [cf. (6)], and observe that it has a number of structural interpretations. Here we are concerned only with those leading to matched filters obeying the general conditions 1) and 2) above. To see how this comes about, let

$$\mathbf{v}_F = \mathbf{Q}_c\mathbf{v} \tag{8}$$

be the discretely filtered (and truncated) wave, obtained on passing the received data $\mathbf{v}$ through a discrete Bayes matched filter $\mathbf{Q}_c = \tilde{\mathbf{Q}}_c = \mathbf{C}^{(1)}$, which also truncates the input to insure the finite samples on $(0, T)$ postulated here. Then we have directly

$$\langle\Phi_n\rangle = \tilde{\mathbf{v}}_F\bar{\mathbf{s}} = \sum_{i=1}^{n} v_F(t_i)(\bar{s}t_i). \tag{9}$$

The result is a discrete cross-correlation without delay (*i.e.*, multiplication) of the filtered input with the locally generated signal $\bar{\mathbf{s}}$. We see accordingly that the linear operation $\mathbf{Q}_c = \mathbf{C}^{(1)}$ $(= \tilde{\mathbf{C}}^{(1)})$ takes the role of a Bayes matched filter, as required here, which may or may not be invariant, and realizable, depending, of course, on the (symmetric) $(n \times n)$ matrix $\mathbf{C}^{(1)}$. Usually, if we invoke invariance we do so at the expense of realizability, while if realizability is possible, we lose invariance.

The matched filter may be expressed in more detail in terms of a (truncated) weighting function h_c:

$$\mathbf{C}^{(1)} = \mathbf{Q}_c = [H_c(t_i\,, t_j)\,\Delta t] = [h_c(t_i - t_j\,, t_i)\,\Delta t], \quad \text{or}$$
$$= [\hat{h}_c(t_i\,; t_i - t_j)\,\Delta t] = \tilde{\mathbf{C}}^{(1)}, \tag{10}$$

since $\mathbf{C}^{(1)}$ is symmetric,[31] exhibiting the time-varying character of the weighting function. Realizability is expressed by the condition

$$H_c(t_i\,, t_j) = h_c(t_i - t_j\,, t_i) = 0,$$
$$t_j > t_i\,, \qquad (\text{or} \quad j > i). \tag{11}$$

However, we cannot impose this condition on $\langle\Phi_n\rangle$ without changing its value, since $v_i s_j C_{ij}^{(1)} \neq c_j s_i C_{ji}^{(1)}$, even if $C_{ij}^{(1)} = C_{ji}^{(1)}$. Accordingly, the time-varying filter h_c (or $\mathbf{Q}_c$) is not realizable here, in consequence of the unsymmetrical character of $\langle\Phi_n\rangle$ itself (in either $\mathbf{v}$ or $\bar{\mathbf{s}}$). This is to be compared with $\langle\Psi_n\rangle$ for incoherent reception (cf. Section III, *B*), where it is possible to have time-varying realizable (Bayes) matched filters (which, on the other hand, do not possess symmetric weighting functions). Note also that if h_c should be invariant, *i.e.*, $h_c(t_i, t_j) = h_c(|\,t_i - t_j\,|)$, realizability is still not possible, because now $h_c \neq 0$, $t_j > t_i$, *i.e.*, because of the postulated symmetry of h_c (or $\mathbf{C}^{(1)}$).

There is, however, another interpretation of the operation $\langle\Phi_n\rangle$, (6), which does make use of a realizable matched filter. To see this, let $\boldsymbol{\gamma}$ be a column vector, such that

$$\mathbf{C}^{(1)}\bar{\mathbf{s}} = \boldsymbol{\gamma}. \tag{12}$$

Then, (9) becomes alternatively

$$\langle\Phi_n\rangle = \tilde{\mathbf{v}}\bar{\boldsymbol{\gamma}} = \sum_{i=1}^{n} v(t_i)\bar{\gamma}_i\,. \tag{13}$$

[29] Here we mean by "realizable" those filters which operate only on the past of the input and, in the case of invariant filters, those whose system functions $Y(i\omega)$ satisfy the Paley-Wiener criterion

$$\left| \int_{-\infty}^{\infty} \frac{\log Y(i\omega)}{1 + \omega^2}\, d\omega \right| < \infty.$$

[30] We use the superscript bar (–) and the brackets $\langle\rangle$ interchangeably to indicate statistical averages over the appropriate random parameters.

[31] One can always write $A(x, y)$ as $C(x - y, x)$, if $A(x, y) = A(y, x)$, *i.e.*, if A is symmetric in its arguments. This is readily demonstrated. Replace y by $-y$ in $A(y, x)$. The result is another functional *form* in x, y, *viz*:

$$A(x, y) = A(y, x) = B(-y, x).$$

The functions are, of course, equal. Whenever $-y$ appears, add x to it, and in addition, add the compensating function(s) of x to maintain equality with the original function. The result is again another functional form in x, y, specifically in $y - x$ and x, *viz*:

$$A(y, x) = B(-y, x) = C(x - y, x),$$

which establishes the desired result. Note that $C(x - y, x) \neq C(x, y - x)$ although A is symmetric.

Next, set

$$\bar{\gamma} = [h_M(T - t_i)\,\Delta t], \tag{14}$$

where h_M is now the weighting function of a time-invariant, realizable linear filter, with a readout at time T $(= n\,\Delta t)$. Eq. (13) becomes

$$\langle \Phi_n \rangle = \sum_{i=1}^{n} v(t_i) h_M(T - t_i)\,\Delta t, \tag{15}$$

so that $\langle \Phi_n \rangle$ is the output of this discrete, realizable, invariant Bayes matched filter at time $t = T$, when $\mathbf{v}$ is the input. In practice, such filters may be achieved in terms of a delay line filter with suitable weighting and readout at $t = T$. From (8)–(10) and (12) it is clear that the Bayes filter $\mathbf{Q}_c$ is related to $\mathbf{h}_M\,\Delta t$ by

$$\bar{\gamma} = \mathbf{Q}_c \bar{\mathbf{s}}, \quad \text{or}$$

$$h_M(T - t_j) = \sum_{i=1}^{n} h_c(t_j - t_i\,,\, t_i)\langle s(t_i\,,\, \boldsymbol{\theta})\rangle. \tag{16}$$

We remark that h_M is closely related to, and in some instances identical with, the S/N matched filters of the earlier theory.[1,2] (Cf. Sections I and IV.)

With continuous sampling the quadratic functional $\langle \Phi_T \rangle$, (7a), which is the continuous analogue of $\langle \Phi_n \rangle$, can be expressed directly as

$$\langle \Phi_T \rangle = \int_{0-}^{T+} v(t) Z_T(t)\,dt = \int_{0-}^{T+} v(t) h_M(T - t)\,dt, \tag{17}$$

where

$$\left.\begin{aligned} Z_T(t) &\equiv \int_{0-}^{T+} G_T^{(1)}(t, u)\langle s(u; \boldsymbol{\theta})\rangle\,du, = 0 \\ &\qquad\qquad \text{outside} \quad (0-, T+) \\ &\equiv h_M(T - t), \end{aligned}\right\} \tag{18}$$

and h_M is once more a (continuous), realizable, invariant matched filter [determined by (18)] with readout at $t = T$. This is the continuous analog of the discrete filter h_M in (14) above. Similarly, the continuous version of (8)–(10) is easily seen to be

$$\langle \Phi_T \rangle = \int_{0-}^{T+} \langle s(t; \boldsymbol{\theta})\rangle v_F(t)\,dt, \tag{19a}$$

with

$$v_F(t) = \int_{0-}^{T+} v(u) H_c(t, u)\,du, \tag{19b}$$

where now h_c and h_M are related by

$$h_M(T - t) = \int_{0-}^{T+} H_c(t, u)\langle s(u; \boldsymbol{\theta})\rangle\,du \tag{20}$$

[cf. (16)]. Again, h_c is not realizable, although it may be invariant if $G_T^{(1)}(t, n) = G_T^{(1)}(|\,t - u\,|)$. Figs. 1 and 2 show the two equivalent realizations of $\langle \Phi_T \rangle$, in terms of H_c and h_M (with obvious modifications in the discrete cases). Usually, h_M will be the easier to instrument than H_c. As a simple example, let us assume a stationary

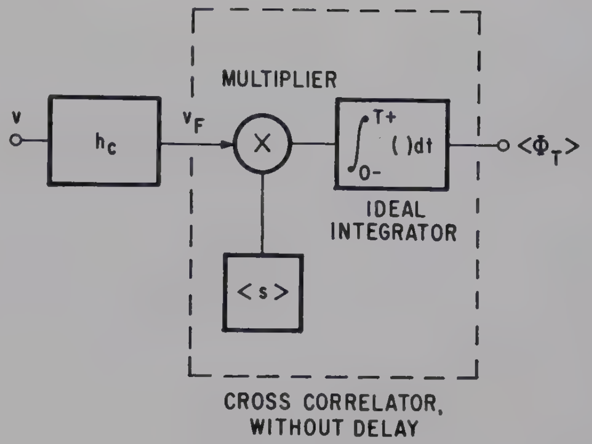

Fig. 1—A nonrealizable Bayes matched filter h_c (coherent reception).

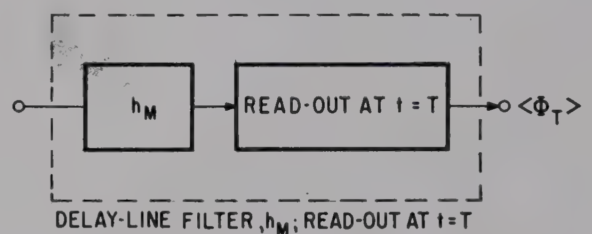

Fig. 2—A time-invariant, realizable Bayes matched filter h_M (coherent reception).

white Gaussian noise background, for which the covariance function is $K_N(t, u) = W_0/2\;\delta(t - u)$. Then, it can be shown that[32] $H_c(t, u) = 2/W_0\;\delta(t - u)$, so that from (20) we get $h_M(T - t) = 2\langle s(t, \boldsymbol{\theta})\rangle/W_0$ on $(0, T)$; $= 0$, elsewhere. This same result also follows from the basic integral equation (63). Consequently, from (19b) we get

$$v_F(t) = 2V(t)/W_0 \quad \text{on} \quad (0, T).$$

The desired number $\langle \Phi_T \rangle$, (19a), becomes

$$\langle \Phi_T \rangle = \frac{2}{W_0}\int_0^T \langle s(t; \boldsymbol{\theta})\rangle^2\,dt \sim \frac{\text{signal energy in } (0, T)}{\text{noise intensity per cycle}},$$

which in the case $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, a set of nonrandom parameters, is again the familiar result of earlier models in this simple situation. (See also the concluding paragraph of Section III.)

B. Incoherent Operation; $\langle \Psi_n \rangle$, $\langle \Psi_T \rangle$

We consider now the considerably more involved situation of incoherent reception, where the fundamental operation on the input data is nonlinear, embodied in the quadratic operations on v in $\langle \Psi_n \rangle$ or $\langle \Psi_T \rangle$ [cf. (6) and (7b)]. As before, we start first with discrete systems, and examine various equivalent classes of realizable, time-varying matched filters which $\langle \Psi \rangle$ contains.

The first class consists of linear filters, which are then followed by zero-memory square-law devices and simple,

[32] Middleton, *op. cit.*, footnote 11, (20.43a) and (20.43b).

or "ideal," integration. As we shall see presently, these filters are time-varying or invariant, realizable or not, in various combinations. Let us begin by introducing a (real) linear discrete filter $\mathbf{Q}$ ($\neq \mathbf{Q}_c$), again with truncation on $(0, T)$, such that

$$\mathbf{v}_F = \mathbf{Q}\mathbf{v}, \tag{21}$$

so that (6) becomes

$$\langle \Psi_n \rangle = \tilde{\mathbf{v}}\overline{\mathbf{C}^{(2)}}\mathbf{v} = \tilde{v}_F\tilde{\mathbf{Q}}^{-1}\overline{\mathbf{C}^{(2)}}\mathbf{Q}^{-1}\mathbf{v}_F > 0, \tag{22}$$

where $\mathbf{v}_F$ is the filter's output. Remembering that $\overline{\mathbf{C}^{(2)}} = \widetilde{\overline{\mathbf{C}^{(2)}}}$, and is positive definite,[33] with $\overline{\mathbf{C}^{(2)}} = 0$, $(i, j > n; i, j < 1)$, since only finite data samples are considered, we note that it is then always possible to find a congruent transformation, $\mathbf{Q}$, which simultaneously diagonalizes $\overline{\mathbf{C}^{(2)}}$ and reduces it to the unit matrix $\mathbf{I}$,[34] *viz*:

$$\left.\begin{aligned}\overline{\mathbf{C}^{(2)}} &= \tilde{\mathbf{Q}}\mathbf{Q} \neq 0, \qquad (1 \le i, j \le n). \\ &= 0 \qquad\qquad\quad \text{elsewhere}\end{aligned}\right\}. \tag{23}$$

Eq. (22) accordingly becomes

$$\langle \Psi_n \rangle = \tilde{\mathbf{v}}_F\mathbf{v}_F = \sum_{j=1}^{n} v_F(t_j)^2, \tag{24}$$

which is just a zero-memory square-law operation on the filtered wave v_{Fj}, followed by an ideal "integration" over the interval $(0, T)$.

Let us list the properties of the discrete filter $\mathbf{Q}$, which is one type of Bayes matched filter:

1) $\mathbf{Q}$ is *linear*. This follows directly from (21).

2) $(\mathbf{Q})_{ij} = 0$, $j > i$. This is simply an expression of the fact that $\overline{\mathbf{C}^{(2)}}$ can be reduced to diagonal form by a congruent transformation with this property. (See Margenan and Murphy,[35] for example.) Accordingly, we observe that $\mathbf{Q}$ represents a *realizable* filter [cf. (11)], since we can write

$$\mathbf{Q} \equiv [H(t_i, t_j)\,\Delta t] = [h(t_i - t_j, t_i)\,\Delta t] = 0,\; t_j > t_i. \tag{25}$$

Note, moreover, that $\mathbf{Q} \neq \tilde{\mathbf{Q}}$, *i.e.*, $H(t_i, t_j) \neq H(t_j, t_i)$, necessarily for realizability.

3) Therefore, $\mathbf{Q} = [h(t_i - t_j, t_i)\,\Delta t]$, all t_i, t_j, is *time-varying*.

It is convenient to set

$$\overline{\mathbf{C}^{(2)}} \equiv [\rho_C(t_i, t_j)\,\Delta t] = [\rho_C(t_j, t_i)\,\Delta t] = \boldsymbol{\varrho}_C\,\Delta t, \tag{26}$$

where we recall from (23) that $\boldsymbol{\varrho}_C = 0$ outside $(1 \le i, j \le n)$, since only operations on data acquired in $(0, T)$ are used in the decision process. The set of basic nonlinear equations determining the elements of $\mathbf{Q}$ can be rewritten

$$\left.\begin{aligned}\rho_C(t_i, t_j) &= \sum_{l=1}^{n} h(t_l - t_i, t_l)h(t_l - t_j, t_l)\,\Delta t, \\ &\qquad\qquad\qquad\qquad (1 \le i, j \le n) \\ &= 0, \qquad\qquad\qquad\qquad \text{elsewhere}\end{aligned}\right\}. \tag{27}$$

The number $\langle \Psi_n \rangle$ becomes accordingly, with the help of (26) and (27) in (22),

$$\langle \Psi_n \rangle = \tilde{\mathbf{v}}\overline{\mathbf{C}^{(2)}}\mathbf{v} = \sum_{j=1}^{n}\left\{\sum_{i=1}^{\infty(\text{or } j)} v(t_i)h(t_j - t_i, t_j)\,\Delta t\right\}^2 = \sum_{j=1}^{n} v_F(t_j)^2, \tag{28}$$

which gives as one interpretation of structure the sequence of a realizable linear time-varying matched filter,[30,36] followed by a zero-memory square-law device and a simple (ideal) "integrator." Note that even if the background noise is stationary [and normal, so that (64a) applies], the time-varying character of h is not removed, basically because of the requirement of a finite sample, on $(0, T)$. For an illustration, see the discussion after (65) below.

Another, equivalent interpretation, also involving a realizable Bayes matched filter, may be found if we now impose invariance on h itself, *i.e.*, require that $h(t_j - t_i, t_j) = h(t_j - t_i)$ in (28), besides realizability. We cannot do this, however, without applying the compensating condition [in (27)] that $h(t_l - t_i) = 0$, $h(t_l - t_j) = 0$, all $t_i, t_j < 0$, or in (28), that $h(t_j - t_i) = 0$, $t_i < 0$. This is equivalent to $h(z_i) = 0$, $z_i > t_l\ (> 0)$, $i = 1, \cdots, n$, and is required by the original condition on $\boldsymbol{\varrho}_C$ that $\boldsymbol{\varrho}_C$ vanish outside $(1, n)$, or $(0, T)$, and that $\tilde{\mathbf{v}}\boldsymbol{\varrho}_C\mathbf{v}$ is kept unchanged. Accordingly, (27) can be reexpressed as

$$\left.\begin{aligned}\rho_C(t_i, t_j) &= \sum_{l=1}^{n} h(t_l - t_i)h(t_l - t_j)\,\Delta t, \\ &\qquad\qquad\qquad\qquad (1 \le i, j \le n) \\ &= 0, \qquad\qquad\qquad\qquad \text{elsewhere}\end{aligned}\right\} \tag{29}$$

and (28) may be rewritten

$$\langle \Psi_n \rangle = \sum_{j=1}^{n}\left\{\sum_{i=1}^{j} h(t_i)v(t_j - t_i)\,\Delta t\right\}^2. \tag{30}$$

Thus, our Bayes matched filter here is now the combination of an invariant, realizable linear filter and a time-varying switch[37] $(j = 1, \cdots, n)$, which "scans" the output of this filter, followed as above, by a zero-memory square-law device and ideal "integration." This is, of course an alternative and equivalent realization

[33] This corresponds physically to the fact that $\langle \Psi_n \rangle$ is a measure of the energy in the received wave during $(0, T)$.

[34] Middleton, *op. cit.*, footnote 11, (20.13) *et seq.*

[35] H. Margenau and G. M. Murphy, "The Mathematics of Physics and Chemistry," D. Van Nostrand Co., Inc., New York, N. Y., sec. 10.6; 1943. See also Middleton, *op. cit.*, footnote 11, footnotes to (20.13) and (20.16b).

[36] Of course, $h(t_j - t_i, t_j) = H(t_j, t_i)$, generally.

[37] One realization of this combination of realizable linear invariant filter and time-varying switch might consist of a delay line or transversal filter [of weighting function h, (29)] whose tap-offs are scanned, or equivalently, readout at a rate equal to that of propagation down the delay line, each readout being applied in sequence to the following zero-memory rectifier. Thus, a typical output is indicated in the braces of (30), at tap-off t_j corresponding to the smoothing period t_j *and* observed at t_j. The time-varying switch consists of the scanning process, which always keeps observation time equal to the integration period, for the different t_j $(0 \le t_j \le T)$.

of the averaged, generalized autocorrelation of the received data v, embodied in $\langle\Psi_n\rangle$, or $\langle\Psi_T\rangle$.

We remark that without the additional feature of a time-varying switch associated with this invariant pre-rectifier filter, it is not possible to write (29) with $\rho_C = 0$, (i, j outside 1, n). In fact, we cannot have this, or any time-invariant filter h alone [which would be equivalent to replacing $i = 1$ by $i \to +\infty$ in the lower limit of the summation in (28)], and still satisfy the condition that ρ_C vanish outside $(1 \le i, j \le n)$, *i.e.*, maintain a finite sample size.[38]

The filters considered so far are realizable. However, it is possible also to represent $\langle\Psi_n\rangle$ in terms of a related subclass of nonrealizable matched filters. This occurs essentially because the reduction of the quadratic form $\tilde{\mathbf{v}}\overline{\mathbf{C}^{(2)}}\mathbf{v}$ to diagonal structure is not uniquely achieved by congruent transformations of the type above, where $(\mathbf{Q})_{ij} = 0$, $j > i$ (the realizability condition). We can also reduce $\langle\Psi_n\rangle$ to a sum of squares, [cf. (24)], by employing another congruent transformation $\mathbf{Q}_0'$ which is a combination of *orthogonal* and diagonal transformations, *viz*:

$$\mathbf{Q}_0' = \mathbf{Q}'\mathbf{Q}_0 \tag{31}$$

where $\mathbf{Q}_0$ is orthogonal, and selected to put $\overline{\mathbf{C}^{(2)}}$ into diagonal form, while $\mathbf{Q}'$ $(= [\lambda_j^{1/2}\,\delta_{ij}])$ reduces the result to the unit matrix $\mathbf{I}$. Here λ_j are the eigenvalues of $\overline{\mathbf{C}^{(2)}}$, which are positive [because of the symmetry and positive definitions of $\overline{\mathbf{C}^{(2)}}$]. The filtered wave is now

$$\mathbf{v}_F' = \mathbf{Q}_0'\mathbf{v}, \tag{32}$$

[cf. (21),] $\mathbf{v}_F' \neq \mathbf{v}_F$. Note that unlike $\mathbf{Q}$ in (21), although $\mathbf{Q}_0'$ is in general, unsymmetrical, it does not vanish $j > i$, so that

$$\mathbf{Q}_0' = [H'(t_i, t_j)\,\Delta t] = [H't_j, t_i)\,\Delta t](\neq 0),$$

which expresses the non-realizability of H' here. In some cases, Q_0' can be symmetrical, as discussed in the footnote examples of footnote 11, pp. 841, 844. The basic relation (23) is now modified to

$$\begin{aligned} \overline{\mathbf{C}^{(2)}} &= \tilde{\mathbf{Q}}_0'\mathbf{Q}_0' \neq 0, \qquad (1 \le i, j \le n) \\ &= 0, \qquad \text{elsewhere} \end{aligned} \Bigg\}, \tag{33}$$

and the analog of (27) becomes[39]

$$\begin{aligned} \rho_C(t_i, t_j) &= \sum_{l=1}^{n} H'(t_l, t_i)H'(t_l, t_j)\,\Delta t, \\ &\qquad\qquad (1 \le i, j \le n) \\ &= 0, \qquad\qquad i, j > n, <1. \end{aligned} \tag{34}$$

[38] In fact, we must have some sort of switch or readout, in any case. We remark also that if Q, (21), were invariant, there would be more equations than unknowns in (23) when $C^{(2)}$ is time-varying. Moreover, if $\overline{C^2}$ were invariant and Q invariant, there would be more unknowns than equations, so that $\overline{C^{(2)}}$ and Q can only be simultaneously invariant or time-varying. In most applications $\overline{C^{(2)}}$ is time-varying (see the examples in Section III, C). For the correction of an error in this regard, see p. 348; this issue.

[39] With $H'(t_l, t_i) = h'(t_l - t_i, t_l) = A(t_l, t_i - t_i)$, etc.

Eq. (28) is thus, alternatively,

$$\langle\Psi_n\rangle = \tilde{\mathbf{v}}_F'\mathbf{v}_F = \sum_{j=1}^{n}\left\{\sum_{i=1}^{n} v(t_i)H'(t_j, t_i)\,\Delta t\right\}^2, \tag{35}$$

with the same type of over-all structure as in the realizable cases above.

The actual weighting functions h, H' in (28), (30), and (35) are determined from the sets of nonlinear equations (27), (29), and (34), respectively. We shall determine their solutions for continuous sampling in Section III, C. Fig. 3 shows a characteristic block structure, with continuous sampling, by an obvious modification of the discrete forms discussed above.

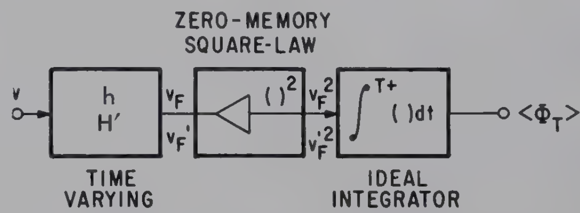

Fig. 3—Interpretation of $\langle\Phi_T\rangle$ in terms of Bayes matched filter of the first kind (incoherent reception).

Our second class of discrete Bayes matched filters consists of a realizable, time-varying, linear filter, which is followed in turn by a zero-memory multiplier and ideal integration to give us still another equivalent interpretation[40] of the quadratic form $\langle\Psi_n\rangle$. Here we use (26) and set

$$\begin{aligned} \overline{\mathbf{C}^{(2)}} \equiv [\rho_C(t_i, t_j)\,\Delta t] &= [\hat{H}(t_i, t_j)\,\Delta t] \\ &= [\hat{h}(t_j - t_i, t_j)\,\Delta t] \end{aligned} \tag{36}$$

where $\hat{h} = \rho_C$ is the weighting function of this time-varying linear filter. Now, in general, if $A_{ij} = A_{ji}$, we can write

$$\sum_{ij}^{n} A_{ij} = 2\sum_{j=1}^{n}\sum_{i=1}^{j} A_{ij}\epsilon_{ij}^{-1}, \qquad \left.\begin{aligned} \epsilon_{ij} &= 1, \quad i \neq j \\ &= 2, \quad i = j \end{aligned}\right\}; \tag{37a}$$

so that $\langle\Psi_n\rangle$, which is a symmetrical form in its components ($A_{ij} = v_iv_jC_{ij}^{(2)}$ here), can be expressed as

$$\begin{aligned} \langle\Psi_n\rangle &= \sum_{ij}^{n} v_iv_j\hat{H}(t_i, t_j)\,\Delta t \\ &= 2\sum_{j=1}^{n} r_j \sum_{i=1}^{j} v_i\hat{h}(t_j - t_i, t_j)\,\Delta t. \end{aligned} \tag{37b}$$

Consequently, the condition of realizability [cf. (25)] can be imposed without changing the value of $\langle\Psi_n\rangle$, since setting $\hat{h}(t_j - t_i, t_j) = 0$, $t_i > t_j$ does not alter the double sum, unlike the earlier situation of coherent reception, where $\langle\Phi_n\rangle$ is not "symmetric" [see comments following (25)]. Thus, if

$$\hat{v}_F = \left[\sum_{i=1}^{j} v(t_i)\hat{h}(t_j - t_i, t_j)\,\Delta t\right] \tag{38}$$

[40] Middleton, *op, cit.*, footnote 11, (20.20) *et seq.*

is the filtered output of this time-varying linear filter,

$$\langle \Psi_n \rangle = 2\tilde{\hat{\mathbf{v}}}_F \mathbf{v} = 2\tilde{\mathbf{v}}\hat{\mathbf{v}}_F = 2 \sum_{i=1}^{n} (\hat{v}_F)_i v_i \tag{39}$$

for (37b), and $\hat{h}$ is realizable, though $\hat{H}$ is symmetric. Since ρ_C is given, $\hat{h}$ is, of course, determined at once from the identity $\rho_C \equiv \hat{h}$, (36) above. Fig. 4 shows the resolution of $\langle \Psi_n \rangle$ in this case (again for the continuous analog $\langle \Psi_T \rangle$).

Finally, we remark that a third class of Bayes matched filters, closely related to and identical in many cases with the S/N matched filters of the earlier theory[1,2,5,6] is sometimes possible (for threshold optimality) in incoherent reception. This occurs if the symmetric "kernel" $\overline{\mathbf{C}^{(2)}}$ can be factored into the matrix product of two vectors, *e.g.*, if

$$\overline{\mathbf{C}^{(2)}} = \bar{\gamma}\tilde{\bar{\gamma}} = [\bar{\gamma}_i \bar{\gamma}_j], \tag{40}$$

where, as in (14),

$$\bar{\gamma} = [h_M(T - t_i)\ \Delta t], \tag{41}$$

and h_M is now the weighting function of a time-invariant, realizable linear filter, with readout at $t = T$, achieved in practice with an appropriate delay-line and readout. [Here, of course, $h_M \neq h_M$ of (14), since $\overline{\mathbf{C}^{(2)}} \neq \mathbf{C}^{(1)}\bar{\mathbf{s}}$, cf. (12).] Then, when (40) holds, we have directly

$$\langle \Psi_n \rangle = (\tilde{\mathbf{v}}\bar{\gamma})(\tilde{\bar{\gamma}}\mathbf{v}) = (\tilde{\mathbf{v}}\bar{\gamma})^2 = \left\{ \sum_{i=1}^{n} v(t_i) h_M(T - t_i)\ \Delta t \right\}^2, \tag{42}$$

which is interpreted as the Bayes matched filter, followed by zero-memory square-law rectification. Note that, unlike the two cases preceeding, there is *no* post rectification integration. Fig. 5 shows system structure for a sample interval $(t_1, t_1 + T)$; since $\bar{\gamma}$ is given [$\overline{\mathbf{C}^{(2)}}$ is known and obeys (40)], h_M follows at once from (41) as before [cf. (16) also]. A brief comment on reception conditions for which we may expect the factorization (40) in practice,

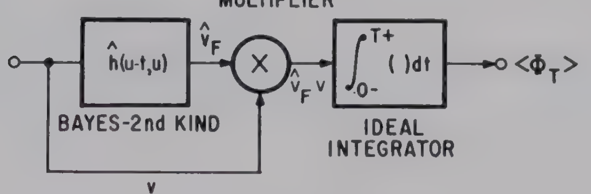

Fig. 4—Resolution of $\langle \Phi_T \rangle$ in terms of Bayes matched filter of the second kind (incoherent reception).

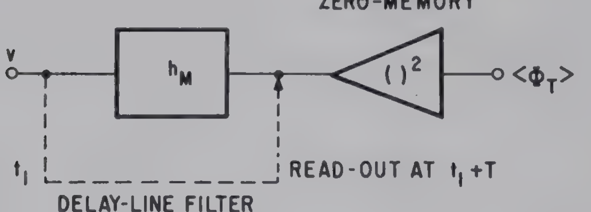

Fig. 5—Evaluation of $\langle \Phi_T \rangle$ by a Bayes matched filter of the third kind (incoherent reception).

is given in Section IV. The combination of h_M and readout at $t = T$, $(t_1 = 0)$, is, of course, a time-varying system, as are all resolutions of $\langle \Psi_n \rangle$ necessarily, since only finite data samples are processed.

There remain the continuous forms of the above. These follow readily from (7b) and the preceeding analysis upon suitable identification of terms. Thus, noting that $\overline{G_T^{(2)}(t, u)}$ may be identified with $\rho_C(t, u)$ where ρ_C is the corresponding continuous form of $\rho_C(t_i, t_j)$ in (26), we see that the basic set of nonlinear equations for the elements $[h(t_i, t_j)\ \Delta t]$ of $\mathbf{Q}$, (21), the first class of Bayes matched filters, goes over into the nonlinear integral equation

$$\left.\begin{aligned} \rho_C(t, u) &= \int_{0-}^{T+} h(x - t, x) h(x - u, x)\, dx, \\ &\qquad\qquad 0- < t, u < T+ \\ &= 0, \qquad\qquad \text{elsewhere} \end{aligned}\right\} \tag{43}$$

where the $(+, -)$ on the regions of existence of ρ_C are inserted to include the total contributions of any possible singularities (*e.g.*, delta functions and their derivatives) that the weighting function h may possess. Then $\langle \Psi_T \rangle$ becomes [as the analog of (28)]

$$\langle \Psi_T \rangle = \int_{0-}^{T+} dt \left\{ \int_0^t v(\tau) h(t - \tau, t)\, d\tau \right\}^2 = \int_{0-}^{T+} v_F(t)^2\, dt, \tag{44a}$$

where

$$v_F(t) = \int_0^t v(\tau) h(t - \tau, t)\, d\tau. \tag{44b}$$

If an invariant filter with time-varying switch is used, this is alternately

$$\langle \Psi_T \rangle = \int_{0-}^{T+} dt \left(\int_{0-}^{t} h(\tau) v(t - \tau)\, d\tau \right)^2 = \int_{0-}^{T+} v_F(t)^2\, dt, \tag{45}$$

where the relation determining h is now the continuous form of (29), *viz*:

$$\left.\begin{aligned} \rho_C(t, u) &= \int_{0-}^{T+} h(x - t) h(x - u)\, dx, \\ &\qquad\qquad (0- < t, u < T+) \\ &= 0, \qquad\qquad \text{elsewhere} \end{aligned}\right\} \tag{46}$$

with $h(x - t) = 0$, $t < 0$, etc. (the switch-condition[41]). The filtered output is

$$v_F(t) = \int_0^t h(\tau) v(t - \tau)\, d\tau, \tag{47}$$

equivalent to (44b).

For the nonrealizable case (31)–(35), the continuous analogues of (32), (34), and (35) are, respectively,

[41] For an error in (47) in an earlier paper (Middleton, *op. cit.*, footnote 21) see p. 348; this issue.

$$\hat{v}_F(t) = \int_{0-}^{T+} H'(t, \tau)v(\tau)\, d\tau, \quad H'(t, \tau) = H'(\tau, t), \tag{48}$$

$$\rho_C(t, u) = \int_{0-}^{T+} H'(x, t)H'(x, u)\, dx,$$
$$(0- < t, u < T+), \tag{49}$$

and

$$\langle \Psi_T \rangle = \int_{0-}^{T+} dt \left\{ \int_{0-}^{T+} v(\tau)H'(t, \tau)\, d\tau \right\}^2. \tag{50}$$

The second Bayes class (for incoherent reception) is described is similar fashion when continuous sampling is employed. The analogs of (38) and (39) are

$$\hat{v}_F(t) = \int_{0-}^{\infty (\text{or } t)} v(t)\hat{h}(t - \tau, t)\, d\tau, \tag{51}$$

and

$$\langle \Psi_T \rangle = 2 \int_{0-}^{T+} \hat{v}_F(t)v(t)\, dt, \tag{52}$$

where $\hat{h}(t - \tau, t) = 0$, $\tau > t$, $(t > 0-)$, for realizability. For the third class of Bayes matched filters, when these are possible [cf. (40)–(42)], we have similarly

$$\langle \psi_T \rangle = \left\{ \int_{0-}^{T+} h_M(T - t)v(t)\, dt \right\}^2, \tag{53}$$

where $h_M(T - t) = 0$, $t > T$, for realizability.

C. Weighting and System Functions of Bayes Matched Filters of the First Kind

There remains the problem of finding the specific weighting functions h, H' in (43) and (49) for the matched filters of the first kind. Let us consider the time-varying case, (43), first, with continuous sampling only. We give a solution originally due to Kailath,[42] and described by him in a letter in this issue, p. 412. We begin by applying a sampling theorem in the frequency domain[43] to h in (43), since $h(x - t, x) = H(x, t)$ is limited by the range of integration in x to $(0, T)$. Moreover, in order that ρ_C vanish in $(0- < t, u < T+)$ we shall also require that $h = 0$, for t outside $(0-, T+)$. This permits us to write

$$\left. \begin{array}{l} H(x, t) = \displaystyle\sum_{-\infty}^{\infty} g_m(t)e^{2\pi mix/T} \\ \qquad (0 \le x \le T);\ = 0 \quad \text{elsewhere} \\ \qquad \text{all } g_m = 0, \quad (t < 0-, > T+,) \end{array} \right\} \tag{54}$$

with a corresponding expression for $H(x, \tau)$, where $\{g_m(t)\}$ are a set of functions to be determined. Substituting into (43) and observing that the integration over x yields $T\, \delta_{mn}$, with δ_{nm} the usual Knonecker delta, we get

$$\rho_C(t, u) = T \sum_{-\infty}^{\infty} g_m(t)g_m(u), \quad 0- < t, u < T+. \tag{55}$$

When ρ_C is symmetrical, positive, definite, and continuous, Mercer's theorem permits[44] the expansion

$$\rho_C(t, u) = \sum_{n=0}^{\infty} \lambda_n\phi_n(t)\phi_n(u), \qquad (0- < t, u < T+) \tag{56}$$

where λ_n are real and nonnegative (some may be zero), and the $\{\phi_n\}$ are the solutions of the homogeneous integral equation[44]

$$\int_0^T \rho_C(t, u)\phi_n(u)\, du = \lambda_n\phi_n(t), \quad (0 \le t \le T). \tag{57}$$

Comparison of (55) and (56) shows that the g_m can be obtained by appropriate relation to the ϕ_n. One such is clearly $g_m(t) = \lambda_n^{1/2}\phi_m(t)/T^{1/2}$, $(0 - < t < T +)$, $= 0$ elsewhere with $g_m(t) = g_{-m}(t)$, since λ_n, ϕ_n are real, in as much as ρ_C is real here, and $n \ge 0$ in (56) and (57). Since the $\{g_m\}$ are not unique, there is considerable freedom of choice[42] for $\{g_m\}$, which is available at the designer's discretion. In any case, the basic structure of this time-varying filter (54), is that of a parallel combination of sequences of time-varying gains (g_m) and realizable, invariant, delay-line filters, $h_m = h_M(T - x) = 2 \cos [2\pi m(T - x)/T] = h_m(x)$, with readouts achieved by suitable tap-offs of the delay line, since (54) can be written (with $g_m = g_{-m}$)

$$H(x, t) = 2 \sum_0^{\infty} g_m(t) \cos (2\pi mx/T) = \sum_0^{\infty} g_m(t)h_m(T - x).$$

For the Bayes matched filter of the first kind in the form of an invariant realizable filter and time-varying switch [cf. (47)], the defining relation is (46), which as Middleton has shown[21,45] can easily be solved by taking an iterated Fourier transform of both sides of (46). The result is

$$| Y(i\omega)_T |^2 = \frac{1}{T} \iint_{-\infty}^{\infty} \overline{G_T^{(2)}(t, u)} e^{-i\omega(t-u)}\, dt\, du$$

$$= \frac{1}{T} \iint_{0-}^{T+} \rho_C(t, u) \cos \omega(t - u)\, dt\, du, \tag{58a}$$

[42] T. Kailath, "Solution of an integral equation occurring in multipath communication problems," to be published in IRE Trans. on Information Theory, vol. IT-6, June, 1960. See also, T. Kailath, "Sampling Models for Linear Time-Variant Filters," M. I. T. Res. Lab. Elec., Cambridge, Mass., Tech. Rept. No. 352; May, 1959.

[43] Middleton, *op. cit.*, footnote 11, sec. 4.2–1, and (4.25) *et seq.*

[44] However, in many cases, $\rho_C(t, u)$ may have singularities in the form of delta functions or their derivatives, so that continuity is no longer available. Then the limits $(0-, T+)$ must be strictly observed, and Mercer's theorem must be appropriately extended to include this more general situation. ρ_C itself in application does not possess singularities of higher order than pairs, *i.e.*, $B\delta(t - 0)\delta(u - 0)$, for instance, so that $\langle \Psi_T \rangle$ is always a finite number (and v is suitably differentiable, if ρ_C contains derivatives of delta functions). But then it can be shown that unsymmetric solutions H' cannot exist. They can only exist when P_C is continuous. Accordingly, we cannot have filters of either type when the signal is deterministic and the background noise is nonwhite (in the additive, gauss case). They are possible, however, with white noise backgrounds. It is in this sense, then, that the example (with symmetric H') in the second footnote of reference 11, p. 857, is to be understood.

[45] Middleton, *op. cit.*, footnote 11, sec. 20.2–2, (20.49) *et seq.* for the reception of deterministic signals.

where Y is the system function corresponding to $h(\tau)$ in (47), *e.g.*,

$$Y(i\omega)_T = \int_{0-}^{\infty} h(\tau)e^{-i\omega\tau}\,d\tau. \tag{58b}$$

An infinite number of these filters is available, since only the modulus of the system function is determined. This occurs in consequence of the condition of sample uncertainty, characteristic of incoherent reception, where precise phase information about the input is unavailable to the observer. Again, this unavoidable lack of uniqueness is, however, an advantage to the system designer, since he is at liberty to select any convenient phase responses $\phi(i\omega)_T$ in $Y_T = |Y_T| e^{i\phi T}$ for the system at hand.

When the nonrealizable filters of (48) and (49) are used instead, we may determine their weighting functions H' by modification of the procedure leading to (56) and (57). In the special case where the H' are symmetric, (54) cannot apply, and the sampling technique of (54) is thus inappropriate. Mercer's theorem[44] suggests that we write[46]

$$H'(x, t) = \sum_m a_m(x)b_m(t),$$
$$0 \le x, t \le T; = 0, \quad \text{elsewhere} \tag{59}$$

and set

$$a_m(x) = \lambda_m^{1/4}\phi_m(x); \qquad b_m(t) = \lambda_m^{1/4}\phi_m(t),$$
$$(0 \le t, x \le T), = 0, \quad \text{elsewhere.} \tag{60}$$

Substituting these relations back in (49) then gives (56), with the $\{\lambda_m\}$, $\{\phi_m\}$ formally obeying (57), as required; H' is accordingly symmetric and time-varying, and it vanishes outside the sampling interval. In this form the time-varying gains and invariant, realizable filters may be interchanged. We remark, finally, that while the weighting functions of the realizable Bayes matched filters of the second and third kind $\hat{h}$, (51) and h_M [when it exists, (53)] are unique. The realizable matched filters of the first kind [h, cf. (44a) and (47)] are not, which gives the system designer an additional degree of freedom in the latter instance. [The nonrealizable version H' [cf. (48)] of the first kind is unique when the solutions of (57) are unique, as is certainly the case when ρ_C is continuous and positive definite in $(0, T)$.]

We conclude this section with a few short examples in the very important situation of additive (though not necessarily stationary) Gaussian noise processes,[47] to show also how the Bayes matched filters of the second kind (coherent reception) [cf. (14) and (18)], and of the third kind (incoherent reception) [cf. (41) and (53)] are frequently identical with the S/N matched filters of the older theory.

For coherent reception it can be shown that[47]

$$\mathbf{C}^{(1)} = \psi^{-1}\mathbf{k}_N^{-1} = \mathbf{K}_N^{-1}, \quad \psi = \overline{N^2}(\bar{N} = 0), \tag{61}$$

so that

$$\mathbf{Q}_c = \mathbf{K}_N^{-1} \quad \text{and} \quad \bar{\boldsymbol{\gamma}} = \mathbf{K}_N^{-1}\bar{\mathbf{s}} = [h_M(t - t_i)\,\Delta t] \tag{62}$$

[cf. (10), (12), (14)]. With continuous sampling the invariant, realizable matched filter h_M is determined from the basic integral equation[48]

$$\int_{0-}^{T+} K_N(t, u)h_M(T - u)\,du = \langle s(t, \boldsymbol{\theta})\rangle,$$
$$0- < t < T+, \tag{63}$$

where $h_M = 0$ outside $(0-, T+)$. But (63) is precisely the determining relation for the S/N matched filter of the earlier theory[49] (where the average over s is omitted). Accordingly, the matched filters in the two theories are structually identical, although the derivation from the decision theory viewpoint is the much more fundamental.

With incoherent reception in normal noise it can be shown similarly that[50]

$$\overline{\mathbf{C}^{(2)}} = \psi^{-2}\mathbf{k}_N^{-1}\overline{\mathbf{s}\tilde{\mathbf{s}}}\mathbf{k}_N^{-1}, \tag{64a}$$

and for continuous sampling,

$$\overline{G^{(2)}(t, u)} = \langle D_T(t, u; \boldsymbol{\theta})\rangle, \tag{64b}$$

where

$$\left.\begin{aligned} D_T(t, u; \boldsymbol{\theta}) &= X_T(t; \boldsymbol{\theta})X_T(u, \boldsymbol{\theta}), \quad 0-, T+) \\ &= 0, \qquad \text{elsewhere} \end{aligned}\right\}. \tag{64c}$$

Here X_T is the solution of the basic integral equation (63), with h_M replaced by X_T and $\bar{\mathbf{s}}$ by s, *viz*:

$$\int_{0-}^{T+} K_N(t, u)X_T(u, \boldsymbol{\theta})\,du = \langle s(t, \boldsymbol{\theta})\rangle \tag{65}$$

and $X_T = 0$ outside $0-$, $T+$, as before.

As a simple example, note that with stationary white Gaussian noise, where $K_N(t, u) = W_0/2\;\delta(t - u)$, we get at once

$$X_T(t, \theta) = \frac{2}{W_0}s(t; \boldsymbol{\theta}); = 0 \quad \text{outside} \quad (0, T). \tag{66}$$

Then, from (64b),

$$\overline{G_T^{(2)}(t, u)} = \frac{4}{W_0^2}\overline{s(t; \boldsymbol{\theta})s(u; \boldsymbol{\theta})} = \rho_C(t, u) \tag{67a}$$

on $(0, T)$, and

$$\langle\Psi_T\rangle = \iint_0^T v(t)\rho_C(t, u)v(u)\,dt\,du. \tag{67b}$$

[46] This solution was achieved independently by Kailath (*op. cit.*, and Middleton (*op. cit.*, footnote 11, sec. 20.2).

[47] Middleton, *op. cit.*, footnote 11, for details see sec. 20.1–1, 20.1–2 in the discrete cases, and sec. 20.2–1, 2 in the continuous cases.

[48] The passage from the discrete to the continuous cases is described in Middleton, *op. cit.*, footnote 11, secs. 19.4–2 and 20.2–1, 2.

[49] Middleton, *op. cit.*, footnote 11, sec. 16.3, particularly (16.99); sec. 16.3–2 for a number of specific examples in the stationary cases; Appendix II for general methods of solution of (63) when $K_N(t, u) = K_N(|t - u|)$, *i.e.*, when the noise is a stationary process.

[50] Middleton, *op. cit.*, footnote 11, sec. 20.1–2.

We may now use (55)–(58) to determine $H(x, t) = h(x - t, x)$, or $h(x - t)$ with a time-varying switch. Thus, from (58) we get for the system function of the invariant h,

$$| Y_T |^2 = \frac{4}{W_0^2 T} \iint_0^T \overline{s(t; \boldsymbol{\theta}) s(u; \boldsymbol{\theta})} \cos \omega(t - u)\, dt\, du, \quad (68)$$

which is evaluated in straightforward fashion, since $s(t; \boldsymbol{\theta})$ is known *a priori*.

Finally, from (64a) and (40) it is clear that unless $\bar{\mathbf{s}}\tilde{\bar{\mathbf{s}}} = \bar{\mathbf{s}} \cdot \tilde{\bar{\mathbf{s}}}$, so that $\overline{\mathbf{C}}^{(2)}$ factors; and, correspondingly in the continuous case, unless $\langle D_T \rangle = \langle X_T(t, \boldsymbol{\theta}) \rangle \langle X_T(u, \boldsymbol{\theta}) \rangle$, it is not possible to obtain a Bayes matched filter of the third kind (incoherent reception), h_M [cf. (63)]. Thus, if h_M [obtained from (63) or (65), on setting $\langle X_T \rangle = h_M(t - u, \boldsymbol{\theta})$ etc.] is actually used, the result is not $\langle \Psi_T \rangle$, but a suboptimum form, which may or may not be close to the optimum value.

Generally, $\langle D_T \rangle$ does not factor, (*i.e.*, $\langle D_T \rangle \not\approx \langle X \rangle \langle X \rangle$, so that the matched filter, h_M, for coherent reception, is not the correct filter for incoherent reception, although in some applications it may be close to the theoretical ideal. Just how close to or distant from the ideal it is depends, of course, on the particular random parameters $\boldsymbol{\theta}$ and their influence on the averages. In radar detection, for example, whenever a potential target may be located anywhere in a large volume, so that the strength of the target return is significantly influenced by distance (here a random parameter of the reception process)[51] we may expect a noticeable difference between h_M (coherent) and the Bayes matched filter h for incoherent reception, with a consequent difference in threshold performance of the detector. (However, see the remarks in the last paragraph of Section IV.)

IV. Concluding Remarks

In the preceeding sections we have shown how the earlier concepts of the matched filter may be generalized by explicitly recognizing the element of decision-making which is characteristic of most reception processes. Accordingly, when the methods of statistical decision theory are used, a variety of new classes of matched filters—the so-called Bayes matched filters—can be defined, and furthermore, can be specifically described in the critical situation of threshold operation, for both detection and extraction. While classification is arbitrary, it is natural to distinguish between the different and equivalent matched filters, first according to the mode of reception, *e.g.*, "coherent" or "incoherent," and second, according to realizability, complexity, and the like. Table I summarizes the results of Section III.

TABLE I
Bayes Matched Filters for Threshold Reception

Type of Reception	Type of Filter	Filter Structure	Structure* of $\left.\begin{matrix}\langle\Phi\rangle\\ \text{or}\\ \langle\Psi\rangle\end{matrix}\right\}$	Remarks
Coherent: $\left\{\begin{aligned}\langle \Phi_n \rangle &= \tilde{\mathbf{v}}\mathbf{C}^{(1)}\bar{\mathbf{s}}\\ \langle \Phi_T \rangle &= \iint_{-\infty}^{\infty} v(t) G_T^{(1)}(t, u) \cdot \langle s(u) \rangle\, dt\, du\end{aligned}\right.$	1). Bayes No. 1.	$H_c(t, \tau) = H_c(\tau, t) = h_c(t - \tau, t)$ Time-varying, non-realizable	h_c + read-out	Unique; specified by $\mathbf{C}^{(1)}$.
	2). Bayes No. 2.	$h_M(T - t)$ Invariant, realizable	h_M + read-out	$\equiv (S/N)$ matched filter; unique; specified by $\mathbf{C}^{(1)}\bar{\mathbf{s}}$, or $G_T^{(1)} \langle s \rangle$.
Incoherent: $\left\{\begin{aligned}\langle \Psi_n \rangle &= \tilde{\mathbf{v}}\overline{\mathbf{C}^{(2)}}\mathbf{v}\\ \langle \Psi_T \rangle &= \iint_{-\infty}^{\infty} v(t) \overline{G_T^{(2)}(t, u)} \cdot v(u)\, dt\, du\end{aligned}\right.$	3). Bayes No. 1.	a). $H(t, \tau) = h(t - \tau, t) \neq H(\tau, t)$ Realizable, time-varying $h(t - \tau, t) = 0, \tau > t.$	a). $h + (\ )^2 + \int_0^T (\ )\, dt$ + read-out	Parallel combinations of: 1). time-varying gains and invariant (realizable) filters, 2). time-invariant, realizable filter with time-varying switch, 1)., 2). not unique.
		b). $H'(t, \tau) = H'(\tau t)$ Nonrealizable, time-varying	b). $H' + (\ )^2 + \int_0^T (\ )\, dt$ + read-out	May or may not be unique. Not possible for deterministic signals and colored noise.
	4). Bayes No. 2.	$\hat{H}(t, \tau) =$ or $\neq$ $\hat{h}(t - \tau, t)$ Realizable, time-varying	$\hat{h} + \otimes + \int_0^T (\ )\, dt$ + read-out	Specified by $\overline{\mathbf{C}^{(2)}}$ or $\overline{G_T^{(2)}}$; unique.
	5). Bayes No. 3.	$h_M(T - \tau)$ Realizable, invariant	$h_M + (\ )^2$ + read-out	Not usually possible, unless $\overline{\mathbf{C}^{(2)}} = \bar{\boldsymbol{\gamma}}\tilde{\bar{\boldsymbol{\gamma}}}$; unique; $\equiv (S/N)$ matched filter.

* "Read-out" $\equiv$ tap-off from appropriate delay line realization, at $t = T$; $(\ )^2 \equiv$ zero-memory, square-law rectification.

[51] Middleton, *op. cit.*, footnote 11, sec. 20.4–6 for a detailed treatment of this radar problem.

Note that in the more usual situation of incoherent reception, these Bayes filters, realizable or not, are all basically time-varying, since we always have to deal with a finite data sample. Moreover, Bayes of the first kind [item 3), Table I] are not unique—there is a degree of freedom for the designer in the details of this structure. (The nonrealizable cases are included, of course, not only for completeness, but because of their potential usefulness in approximation.)

With coherent reception an optimum, realizable, invariant filter identical to that based on maximization of signal-to-noise ratio (the S/N matched filter) is possible, while for incoherent reception such an identity does not ordinarily occur. The optimum, "matched" filters are Bayes Type 1 or Type 2 [items 3) and 4) in Table I], so that the S/N matched filter is not optimum here. We remark, however, that when a structural identity between Bayes matched filters of Type 2 (coherent reception) or Type 3 (incoherent reception), and the S/N matched filters of the earlier theory does occur, it is not an inherent property of the matched filter concept. Rather, it is the result of 1) the particular signal and noise statistics involved, and 2) the fact that in the threshold Bayes case and in the S/N formulation (regardless of signal strength *vis-à-vis* the noise) the respective optimizations are both based on quadratic forms involving the effective signal and noise energies, so that enhancement[52] of signal vs noise in either instance, although for quite different criteria, lead to the same matched filters. For example, this identity effectively occurs when the received signal is narrow band, and when (if the signal is present) only the amplitude and RF phase are unknown at the receiver. Note however, that the Bayes matched filters are the result of an actual decision process, minimizing average cost or risk, where all pertinent knowledge (and ignorance) has been systematically employed in the formulation of optimality, while the S/N formulation is based on the more limited and incomplete second-moment criterion of signal-to-noise ratio. For these reasons, the Bayes matched filter is the much broader concept, subsuming the S/N matched filter, just as second-moment operations (*e.g.* calculation of spectra, covariance functions) are subsummed by those using the statistically more complete information contained in the nth-order distribution ($n \rightarrow \infty$). Also, for these reasons, as Section III above has already explicitly indicated, the structure of Bayes matched filters may differ considerably from those based on the more elementary criterion.

Finally, for strong signals it is clear that although the matched filtering and subsequent linear and nonlinear operations described here for threshold systems are no longer optimum at these higher signal levels, they are absolutely better than in the threshold situation, and hence usually satisfactory. Thus, if the designer strives for threshold optimality, he will achieve effective performance at higher levels, as long as he takes care to avoid any destruction of desired information through saturation of one kind or another.

[52] Middleton, *op. cit.*, footnote 11, sec. 20.2–4 for a geometric interpretation.

Correlation Detection of Signals Perturbed by a Random Channel*

THOMAS KAILATH†

Summary—**We show that the concept of correlation detection deterministic signals in additive Gaussian noise can be extended a natural manner to the detection of signals that are transmitted rough a "Gaussian" random channel besides being corrupted by lditive Gaussian noise. Such situations are typical in communica- on over scatter-multipath channels (with or without a specular mponent). In the deterministic case, the receiver essentially osscorrelates the received signal with the signal before the lditive noise was introduced. When a random channel is present, wever, this latter signal, *i.e.*, the output of the random channel, is t known at the receiver, However, knowing the statistics of the annel and the noise, the receiver can make an estimate of it from e received signal on the hypothesis that a particular signal was ansmitted. The optimum receiver then crosscorrelates this esti- ate with the received signal.**

I. Introduction

THE detection of deterministic signals (*i.e.*, signals whose form is exactly known at the receiver) in white Gaussian noise has been studied by a number authors—notably North,[1] Van Vleck and Middleton,[2] ıd Woodward.[3] They found that the operation per- rmed by an "optimum" receiver is essentially a cross- rrelation, as shown in Fig. 1. The receiver crosscorre- tes the received signal, say $y(t)$, with each possible ansmitted signal $x^{(k)}(t)$. This operation can be per- rmed by a multiplier-integrator combination, with read- ıt at time T, where T is the duration of the $x^{(k)}(t)$, by a "matched filter" with readout at T. The term ptimum" merits some explanation: Woodward has own that all of the information in the received signal t) is contained in the set of *a posteriori* probabilities $x^{(k)}(t) \mid y(t)\}$. These probabilities can then be weighted ıd combined according to different criteria—for example, eyman-Pearson, Ideal observer, etc.—in order to make e final decision as to which signal $x^{(k)}(t)$ was actually esent. In Fig. 1, the box marked D denotes this final processing, while the outputs of the crosscorrelators are simply related to the *a posteriori* probabilities. In this paper, we shall only consider the problem of obtaining these probabilities, and this will define our "optimum" receiver.

Our purpose is to determine optimum receivers to detect signals that have been transmitted through a channel A, and have been corrupted by additive Gaussian noise $n(t)$ as shown in Fig. 2. Now if channel A is by some means exactly known at the receiver, the problem can be solved by a trivial modification of the receiver of Fig. 1. That is, instead of crosscorrelating $y(t)$ and $x^{(k)}(t)$, we crosscorrelate $y(t)$ and $z^{(k)}(t)$, the channel output signal for a particular input $x^{(k)}(t)$ (*cf.* Fig. 3). The

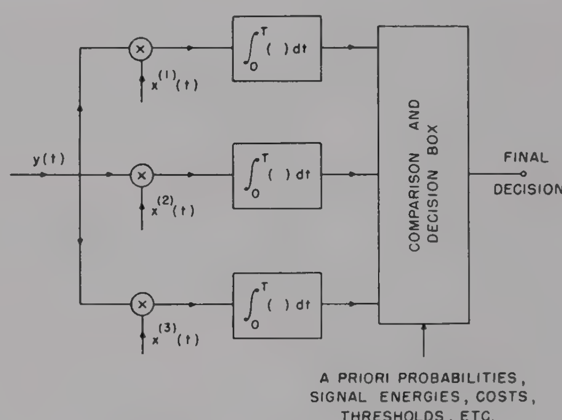

Fig. 1—A functional form of optimum receiver for detection of deterministic signals in white Gaussian noise.

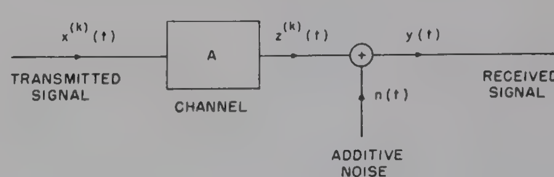

Fig. 2—The channel.

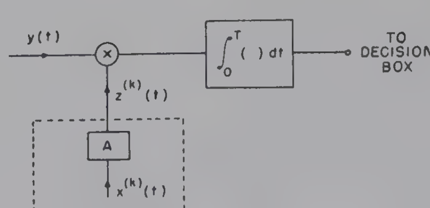

Fig. 3—An element of the optimum receiver for the case of a signal perturbed by a known channel A.

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[1] D. O. North, "An Analysis of the Factors which Determine gnal/Noise Discrimination in Pulsed-Carrier Systems," RCA bs., Princeton, N. J., Tech. Rept. PTR-6C; 1943.

[2] J. H. Van Vleck and D. A. Middleton, "Theoretical comparison the visual, aural, and meter reception of pulsed signals in the esence of noise," *J. Appl. Phys.*, vol. 17, pp. 940–971; November, 46.

[3] P. M. Woodward, "Probability and Information Theory with plications to Radar," Pergamon Press, London, Eng.; 1953.

signal $z^{(k)}(t)$ can be obtained by passing $x^{(k)}(t)$ through the (known) filter A. (We shall use the words filter and channel interchangeably.) In our problem, however, we shall assume that filter A is random and therefore not known exactly at the receiver. Of course, we need further assumptions on the filter in order to solve the problem, and these are discussed in detail in the next section.

But briefly, if we assume that channel A is such that its output $z^{(k)}(t)$ for an input $x^{(k)}(t)$ is Gaussianly distributed with known (zero) mean and variance, (*cf.* the detailed discussion in Section III) then the optimum receiver first makes a best possible estimate of $z^{(k)}(t)$, say $z_e^{(k)}(t)$—we assume that $x^{(k)}(t)$ was sent—and then crosscorrelates $y(t)$ and $z_e^{(k)}(t)$ to produce an output simply related to $p\{x^{(k)}(t) \mid y(t)\}$.

A word about the mathematics used in this paper is in order. All we need is some rather elementary matrix algebra. However, the formulation used here is not always the most general or the most elegant—such an approach is unnecessary for the specific problem discussed, but is quite valuable and proper in a more general study, which we hope to publish at a future date.

II. Definition of the Problem

The general problem is the following: one of a finite set of known signals $\{x^{(k)}(t)\}$ of limited duration is transmitted through a random linear time-variant channel A of finite memory, resulting in a waveform, say $z^{(k)}(t)$, which is further corrupted by additive noise, say $n(t)$, before being available to the receiver (*cf.* Fig. 2). Let $y(t)$ denote the final received signal, that is, $y(t) = n(t) + z^{(k)}(t)$, and let T denote the duration of $y(t)$. We then define the optimum receiver in the sense of Woodward[1] as being one that computes the *a posteriori* probabilities $p\{x^{(k)}(t) \mid y(t)\}$.

We assume that the additive noise is Gaussian (though not necessarily white) and that the random channel is such that its output $z^{(k)}(t)$, for a particular input $x^{(k)}(t)$, is Gaussian. We also assume that the parameters (the mean and the variance) of these distributions are known *a priori*; however, the distributions themselves need not be stationary. No further assumption is made about the structure of the channel; that is, whether it consists of a finite number of paths, independent or otherwise, or whether there is only a continuum of paths, etc. Channels with a number of relatively delayed, randomly-varying paths, the so-called "scatter-multipath" channels, are often of this type, and our model includes the cases of Rayleigh fading, and of fading when a specular component is also present (so-called "Rician" fading), with arbitrary rates of variation.

In this paper we shall make a few additional assumptions; namely, that $n(t)$ has zero mean, and also that $z^{(k)}(t)$ and $n(t)$ are independent, for all k. Again we shall initially derive the result for a low-pass (video) channel, and then indicate how the analysis should be modified for a band-pass channel.

III. A Model for the Channel

We shall find it convenient to use a discrete analysis in this work. While it is true that we might work directly with continuous functions, the approach is then mathematically more involved because it uses integral equations in place of the matrix algebra used in the discrete case. The physical significance of the steps is often obscured by the mathematics required to manipulate the integral equations correctly, whereas the elementary matrix algebra used in this paper should be relatively familiar to the reader. The author has performed the analysis with integral equations but has not found any advantages in the method.

As a first step, we shall consider the discrete analog of the convolution integral that relates the input and output of the channel filter A. The convolution formula gives

$$z(t) = \int_0^t a(\tau, t)x(t - \tau)\, d\tau, \tag{1}$$

where

$a(\tau, t)$ = impulse response of the filter
= response of filter measured at time t to an impulse input τ seconds ago; we shall also assume $a(\tau, t) = 0$, $\tau < 0$; *i.e.*, the filter is physically realizable,

$x(t)$ = input to the filter; $x(t) = 0$, $t < 0$.

We would expect the discrete analog of this equation to be

$$z(m) = \sum_{k=0}^{m} a(k, m)x(m - k), \tag{2}$$

or

$$\mathbf{z} = \mathbf{A}\mathbf{x} \tag{3}$$

where m, k are integers (on a suitably normalized time scale) and $\mathbf{x}_t = \lfloor x(0)x(1)x(2) \cdots \rfloor$ is a row vector representing the discrete input signal $x^{(k)}$; the subscript t denotes the transposition of the matrix; $\mathbf{z}$ similarly represents the output waveform; and A represents the channel. To get a feeling for this formula, consider the discrete channel shown in Fig. 4. The reader can easily verify the following matrix relation:

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} a_0(0) & 0 & 0 \\ a_1(1) & a_0(1) & 0 \\ a_2(2) & a_1(2) & a_0(2) \\ 0 & a_2(3) & a_1(3) \\ 0 & 0 & a_2(4) \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{00} & 0 & 0 \\ a_{11} & a_{01} & 0 \\ a_{22} & a_{12} & a_{02} \\ 0 & a_{23} & a_{13} \\ 0 & 0 & a_{24} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}. \tag{4}$$

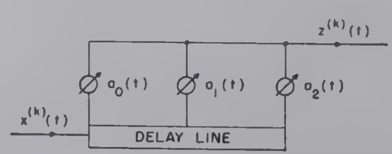

Fig. 4—A simple delay-line channel.

We have written for convenience $a(k, m) = a_k(m) = a_{km}$. With reference to Fig. 4, notice that for fixed k, we can interpret a_{km} as the values assumed by the kth tap-function. (Notice that the number of *columns* in A depends on the duration of x and that the number of rows depends on the sum of the channel memory and the duration of x. Therefore A is rectangular, unless we have a single-path channel.) In Kailath[4] it is shown that, in general, an arbitrary linear time-variant filter can be represented as closely as we wish by a discrete filter of the type shown in Fig. 4, for which a matrix equation like (3) always holds. We shall therefore assume a matrix input-output equation in future work.

To make our channels' models Gaussianly random, we shall assume that the "tap-functions" $a_i(t)$ on the delay line are sample functions from a Gaussian process. We shall assume that the $a_i(t)$ have zero means and that the correlation matrix

$$\mathbf{\Phi}_{AA} = \begin{bmatrix} \overline{a_{00}^2} & \overline{a_{00}a_{01}} & \cdots & \overline{a_{00}a_{11}} & \cdots & \overline{a_{00}a_{22}} & \cdots \\ \overline{a_{01}a_{00}} & \overline{a_{01}^2} & \cdots & \cdot & \cdots & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{a_{11}a_{00}} & \overline{a_{11}a_{01}} & \cdots & \overline{a_{11}^2} & \cdots & \overline{a_{11}a_{22}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{a_{22}a_{00}} & \overline{a_{22}a_{01}} & \cdots & \overline{a_{22}a_{11}} & \cdots & \overline{a_{22}^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{5}$$

is known. ($\mathbf{\Phi}_{AA}$ is, of course, symmetrical.)

With such a channel, the output signal $\mathbf{z}^{(k)}$ for a given input signal $\mathbf{x}^{(k)}$ is Gaussianly distributed with zero mean and covariance matrix

$$\mathbf{\Phi}_{zz}^{(k)} = [\phi_{ij}^{(k)}] \tag{6}$$

where $\phi_{ij}^{(k)} = \overline{z_i^{(k)} z_j^{(k)}}$. A formalized method of calculating $\mathbf{\Phi}_{zz}^{(k)}$ given $\mathbf{x}^{(k)}$ and $\mathbf{\Phi}_{AA}$ can be developed, but is not needed here. Instead, we shall illustrate the calculation for the simple channel of Fig. 4. There we have

$$\begin{aligned} \overline{z_0^2} &= \overline{a_{00}^2}x_0^2\,, \\ \overline{z_0 z_1} &= \overline{a_{00}a_{01}}x_0x_1 + \overline{a_{00}a_{11}}x_0^2\,, \\ \overline{z_1 z_2} &= \overline{a_{11}a_{22}}x_0^2 + (\overline{a_{11}a_{12}} + \overline{a_{01}a_{22}})x_0x_1 + \overline{a_{11}a_{02}}x_0x_2 \\ &\qquad + \overline{a_{01}a_{02}}x_1x_2 + \overline{a_{01}a_{12}}x_0^2\,, \end{aligned} \tag{7}$$

and so on. And finally, if $\mathbf{z}$ (for a given $\mathbf{x}^{(k)}$) is Gaussian, and the additive noise $\mathbf{n}$ is Gaussian, the received signal

$$\mathbf{y} = \mathbf{A}\mathbf{x}^{(k)} + \mathbf{n} = \mathbf{z}^{(k)} + \mathbf{n} \tag{8}$$

is also Gaussian and we may write

$$p(\mathbf{y} \mid \mathbf{x}^{(k)}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot \frac{1}{|\mathbf{\Phi}_{yy}^{(k)}|^{1/2}} \cdot \exp\left[-1/2\{\mathbf{y}_t[\mathbf{\Phi}_{yy}^{(k)}]^{-1}\mathbf{y}\}\right], \tag{9}$$

where $\mathbf{\Phi}_{yy}^{(k)}$ is the covariance matrix of $\mathbf{y}$ (given $\mathbf{x}^{(k)}$). If we assume the additive noise to be statistically independent of the channel, then

$$\mathbf{\Phi}_{yy}^{(k)} = \mathbf{\Phi}_{zz}^{(k)} + \mathbf{\Phi}_{nn}\,. \tag{10}$$

We now have all of the relations that we need for the study of the optimum receiver itself.

IV. The Optimum Receiver

As stated in Section I, the optimum receiver computes the set of *a posteriori* probabilities $\{p(\mathbf{x}^{(k)} \mid \mathbf{y})\}$. We shall show how to get one of these, say $p(\mathbf{x}^{(k)} \mid \mathbf{y})$.

If we use Bayes' rule (or theorem), we can write

$$p(\mathbf{x}^{(k)} \mid \mathbf{y}) = \frac{p(\mathbf{y} \mid \mathbf{x}^{(k)})p(\mathbf{x}^{(k)})}{\sum_k p(\mathbf{y} \mid \mathbf{x}^{(k)})p(\mathbf{x}^{(k)})}. \tag{11}$$

We shall assume that the $p(\mathbf{x}^k)$ are known; then, since the denominator of (11) is a constant, what we essentially have to compute is the "forward" probability $p(\mathbf{y} \mid \mathbf{x}^{(k)})$. This is given by the multivariate Gaussian distribution as in (9). But notice that since the exponential is a single-valued function of its argument, we might just as well compute the quadratic form

$$\Lambda'^{(k)} = \mathbf{y}_t[\mathbf{\Phi}_{yy}^{(k)}]^{-1}\mathbf{y}. \tag{12}$$

Furthermore, we can write [as shown in Appendix I, (32)]

$$\Lambda'^{(k)} = \mathbf{y}_t\mathbf{\Phi}_{nn}^{-1}\mathbf{y} - \mathbf{y}_t\mathbf{\Phi}_{nn}^{-1}\mathbf{H}^{(k)}\mathbf{y}, \tag{13}$$

where

$$\mathbf{H}^{(k)} = \mathbf{\Phi}_{zz}^{(k)}[\mathbf{\Phi}_{yy}^{(k)}]^{-1} = \mathbf{I} - \mathbf{\Phi}_{nn}[\mathbf{\Phi}_{yy}^{(k)}]^{-1}. \tag{14}$$

Since the first term on the right-hand side of (13) is independent of $x^{(k)}$, we need only to consider the second term:

$$\Lambda^{(k)} = \mathbf{y}_t\mathbf{\Phi}_{nn}^{-1}\mathbf{H}^{(k)}\mathbf{y} \tag{15}$$

and $\Lambda^{(k)}$, which is also a quadratic form, will determine the operations that the receiver will have to perform. These we shall now consider.

It is instructive to examine first the case in which the additive noise is white Gaussian, with a noise power of 1 watt per cycle; that is, $\mathbf{\Phi}_{nn} = \mathbf{I}$, the identity matrix. Then we have

$$\Lambda^{(k)} = \mathbf{y}_t\mathbf{H}^{(k)}\mathbf{y} = (\mathbf{y})_t(\mathbf{H}^{(k)}\mathbf{y}). \tag{16}$$

To compute this, we can use the receiver structure that is shown in Fig. 5; that is, we pass $\mathbf{y}$ through the filter $\mathbf{H}$, then multiply the output of $\mathbf{H}$ by $\mathbf{y}$, and integrate for a time T. The output of the integrator at time T is simply related to $p(\mathbf{y} \mid \mathbf{x}^{(k)})$; the multiplier and integrator combination is just a crosscorrelator. But what is the role

[4] T. Kailath, "Sampling Models for Linear Time-Variant Filters," Res. Lab. of Electronics, M. I. T., Cambridge, Mass., Tech. Rept. 52; May, 1959.

of the filter $\mathbf{H}^{(k)}$? It is shown in Appendix I that $\mathbf{H}^{(k)}$ defined as in (14) is a filter that operates on $\mathbf{y}\ (= \mathbf{z}^{(k)} + \mathbf{n})$ to give an estimate, $\mathbf{z}_e^{(k)}$, say, of $\mathbf{z}$.

This estimate is a minimum average mean-square error (or Wiener) estimate, which is equivalent to a maximum likelihood estimate for our situation, where we have Gaussian statistics. The filters $\mathbf{H}$ are, in general, unrealizable. When we have white Gaussian noise, however, the $\mathbf{H}^{(k)}$ are symmetric, and therefore we can write

$$\Lambda^{(k)} = \mathbf{y}_t \mathbf{H}'^{(k)} \mathbf{y}, \tag{17}$$

where $\mathbf{H}'^{(k)}$ is a realizable filter. $\mathbf{H}'^{(k)}$ is obtained from $\mathbf{H}^{(k)}$ by omitting all terms above the main diagonal and doubling all terms below the main diagonal. This is proved in Appendix I. Therefore in Fig. 5, the output of the filter $h(\tau, t)$ (which is a realizable filter) is $z_e^{(k)}$, and hence the receiver effectively crosscorrelates $y(t)$ and $z_e^{(k)}(t)$ to give an output directly related to $p(x^{(k)} \mid y)$. Thus we have a rather natural generalization of the situation in which the channel A is known, for which case an optimum receiver is shown in Fig. 3. Finally, note that we can also set up a matched filter type of receiver to perform the crosscorrelation in (16).

Now let us return to the case in which the noise is not white and therefore $\mathbf{\Phi}_{nn}^{-1} \neq \mathbf{I}$. In this case,

$$\Lambda^{(k)} = \mathbf{y}_t \mathbf{\Phi}_{nn}^{-1} \mathbf{H}^{(k)} \mathbf{y}. \tag{18}$$

Here, a receiver structure of the type shown in Fig. 5 can be used. Since $\mathbf{\Phi}_{nn}^{-1}\mathbf{H}^{(k)}$ is symmetric [it is equal to the difference of two symmetric matrices, *cf.* (14)], we can again use the method given above, *cf.* (17), to make the filter $\mathbf{\Phi}_{nn}^{-1}\mathbf{H}^{(k)}$ realizable. Note that while $\mathbf{\Phi}_{nn}^{-1}$ is symmetric, $\mathbf{H}^{(k)}$ is not necessarily symmetric. In this receiver, $\mathbf{H}^{(k)}$ again plays the role of a maximum likelihood filter that produces as its output the best possible estimate of $\mathbf{z}^{(k)}$, say $\mathbf{z}_e^{(k)}$.

While we thus have a receiver structure for the nonwhite noise case, we can obtain another structure which is equivalent to this, but has a direct physical interpretation. To do so, we note that there is a theorem in matrix algebra[5] which states that a positive-definite matrix can be factored into the product of a triangular matrix and its transposition. Applying this theorem to $\mathbf{\Phi}_{nn}^{-1}$, we can write

$$\mathbf{\Phi}_{nn}^{-1} = \mathbf{W}_t \mathbf{W}, \tag{19}$$

where $\mathbf{W}$ is a matrix with all zeros above the main diagonal. Using (19), we can write

$$\begin{aligned}\Lambda^{(k)} &= \mathbf{y}_t \mathbf{W}_t \mathbf{W} \mathbf{H}^{(k)} \mathbf{y} \\ &= (\mathbf{y}\mathbf{W})_t (\mathbf{W}\mathbf{H}_y^{(k)}).\end{aligned} \tag{20}$$

A receiver structure for implementing (20) is shown in Fig. 6. Now it is proven in Appendix II that the filter $\mathbf{W}$ acts as a "whitening" filter for the noise component of the received signal $\mathbf{y}$. The action of the receiver is now clear: to reduce the nonwhite case to the white noise case, we pass $\mathbf{y}$ through a "whitening" filter $\mathbf{W}$. However, $\mathbf{W}$ distorts the signal portion $\mathbf{z}^{(k)}$ of $\mathbf{y}$, and to compensate for this distortion, we pass $\mathbf{z}_e^{(k)}$ (which is the output of the filter $\mathbf{H}^{(k)}$) through the same filter $\mathbf{W}$. We are now in the same situation as in the case for white noise, and the optimum operation, once again, is the crosscorrelation of $\mathbf{y}$ and $\mathbf{z}_e^{(k)}$.

Thus we see in Figs. 5 and 7 how the concept of correlation detection extends in a rather natural fashion to the case of transmission through a random channel. However, the extension is incomplete in one respect; we cannot now assume that A is not random and get the result shown in Fig. 3. This is because, here, $\mathbf{z}$ is no longer random and $\mathbf{y}$ is not a Gaussian distribution with *zero* mean; therefore (9) is not true. But the situation is easily remedied. We shall assume that channel A is such that $\mathbf{z}^{(k)}$ is Gaussian with a mean $\bar{\mathbf{z}}^{(k)}$ which is known *a priori*. Then $\mathbf{z}^{(k)}$ can be written as the sum of a random component $\mathbf{z}_r^{(k)}$ and a constant component $\bar{\mathbf{z}}^{(k)}$:

$$\mathbf{z}^{(k)} = \mathbf{z}_r^{(k)} + \bar{\mathbf{z}}^{(k)} \tag{21}$$

and similarly, we can write

$$\mathbf{z}^{(k)} = \mathbf{A}\mathbf{x}^{(k)} = \mathbf{A}\mathbf{x}^{(k)} + \bar{\mathbf{A}}\mathbf{x}^{(k)}. \tag{22}$$

With these additional assumptions and definitions, it can readily be seen that $\mathbf{y}$ is Gaussianly distributed with mean $\bar{\mathbf{z}}$ and covariance matrix

$$\mathbf{\Phi}_{z_r z_r}^{(k)} + \mathbf{\Phi}_{nn} = \mathbf{\Phi}_{yy}^{(k)}. \tag{23}$$

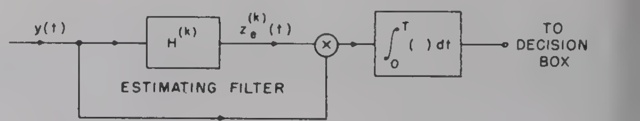

Fig. 5—An element of the optimum receiver for a random channel A (for white Gaussian noise).

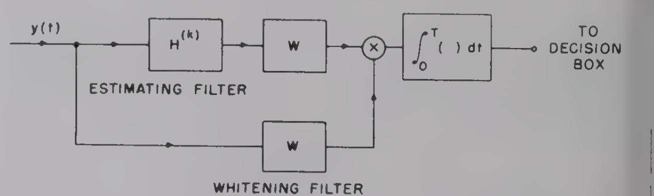

Fig. 6—An element of the optimum receiver for a random channel A (for nonwhite Gaussian noise).

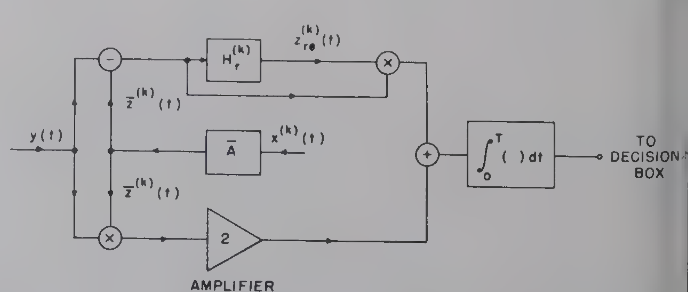

Fig. 7—An element of the optimum receiver for a specular plus a random component and white Gaussian noise.

[5] E. A. Guillemin, "The Mathematics of Circuit Analysis," John Wiley and Sons, Inc., New York, N. Y.; 1949.

Since we have already gone through the derivations in detail for the case where $\bar{\mathbf{z}}^{(k)}$ is zero, we can proceed by analogy without much further explanation.

Thus,

$$\Lambda'^{(k)} = (\mathbf{y} - \bar{\mathbf{z}}^{(k)})_t[\mathbf{\Phi}_{yy}^{(k)}]^{-1}(\mathbf{y} - \bar{\mathbf{z}}^{(k)}) \tag{24}$$

$$= (\mathbf{y} - \bar{\mathbf{z}}^{(k)})_t\mathbf{\Phi}_{nn}^{-1}(\mathbf{y} - \bar{\mathbf{z}}^{(k)}) - (\mathbf{y} - \bar{\mathbf{z}}^{(k)})_t\mathbf{\Phi}_{nn}^{-1}\mathbf{H}_r^{(k)}(\mathbf{y} - \bar{\mathbf{z}}^{(k)}), \tag{25}$$

where

$$\mathbf{H}_r^{(k)} = \mathbf{\Phi}_{z_r z_r}^{(k)}[\mathbf{\Phi}_{z_r z_r}^{(k)} + \mathbf{\Phi}_{nn}]^{-1} \tag{26}$$

and then, if we assume that the transmitted signals $x^{(k)}$ all have the same energy, and that the noise is white, with power of 1 watt per cps, we have

$$\Lambda^{(k)} = +2\mathbf{z}_t^{(k)}\mathbf{y} + (\mathbf{y} - \mathbf{z}^{(k)})_t\mathbf{H}_r^{(k)}(\mathbf{y} - \mathbf{z}^{(k)}). \tag{27}$$

$\Lambda^{(k)}$ determines the receiver structure which is shown in Fig. 7.

Now notice that if $\mathbf{z}_r^{(k)} = 0$ — that is, there are no random components in the channel—we have the receiver that is shown in Fig. 3. Modifications can be made as before when the noise is not white and/or when the transmitted signals are not of equal energy.

One final point should be made. Our analysis has assumed low-pass (video) signals and channel. The extension to the band-pass case is easily made. Consider the row matrix $\mathbf{x}_t^{(k)}$ to be in the form

$$\mathbf{x}_t^{(k)} = \lfloor \mathbf{x} \vdots \hat{\mathbf{x}} \rfloor = \lfloor x_0 x_1 \cdots \vdots \hat{x}_0 \hat{x}_1 \cdots \rfloor,$$

where the $\hat{x}_i^{(k)}$ are samples from the Hilbert transform $\hat{x}(t)$ of $x(t)$. We write the channel matrix A in similar partitioned form so that

$$\mathbf{z}] = \frac{\lfloor \mathbf{A} \vdots -\hat{\mathbf{A}} \rfloor}{} \left.\begin{matrix}\mathbf{x}\\ \cdots \\ \hat{\mathbf{x}}\end{matrix}\right]$$

This, however, is still of the form

$$\mathbf{z} = \mathbf{A}\mathbf{x}$$

as in (3) (where $\mathbf{z}$, $\mathbf{x}$, $\mathbf{A}$ are now partitioned matrices), and therefore all of our analysis for the low-pass case can be carried over directly to the band-pass case. The receiver structures must be modified suitably, but the changes are readily made and we shall not discuss them here.

V. Concluding Remarks

We have shown in Figs. 5 and 7 that the optimum receiver for detection of signals transmitted through a random Gaussian channel can be interpreted as a bank of "estimator-correlator" combinations. A result of this type was originally obtained by Price,[6] with a somewhat different interpretation, for a channel that consisted of a single scatter path without a specular component and with white additive Gaussian noise. In Price's receiver, to test the hypothesis that a particular signal, say $x^{(k)}(t)$, was transmitted, an estimate is first made, from the received signal $y(t)$, of the path values, assuming that $x^{(k)}(t)$ was transmitted. [Note that unless in fact $x^{(k)}(t)$ was actually the transmitted signal, this estimate is not an estimate of the true channel path.] This estimate is then crosscorrelated with the product of $x^{(k)}(t)$ and $y(t)$ to give a number simply related to $p(x^{(k)} \mid y)$. This receiver can be reinterpreted in terms of our structure in Fig. 5. However, our interpretation and the matrix analysis, first, provide a much simpler proof of Price's result and second, enable the proof to be extended to more general channels, while also revealing the receiver as a natural generalization of the deterministic channel case. This receiver structure is very intuitively satisfying and we might reasonably expect it to be good even in situations that do not quite conform to our theoretical model. It can also be shown that similar structures are optimum for arbitrary statistics, provided that the noise power is very high; *i.e.*, we are operating with low "signal-to-noise" ratios.

Finally, we might point out that the receiver structures described here are not the only ones possible. We have found some other forms with equally interesting physical interpretations. (It is hoped that further results will be published at a later date; they can currently be found in preliminary form.[7]

Appendix I
The Minimum Variance Estimator

We have $\mathbf{y} = \mathbf{z} + \mathbf{n}$ and we want to find the linear filter, $\mathbf{H}$, not necessarily realizable, for which

$$\mathbf{z}_e = \mathbf{H}\mathbf{y} \tag{28}$$

and

$$\boldsymbol{\varepsilon} = \overline{(\mathbf{z} - \mathbf{z}_e)^2} = \text{a minimum}. \tag{29}$$

$\mathbf{y}$, $\mathbf{n}$, $\mathbf{z}$, $\boldsymbol{\varepsilon}$ are all column matrices of length proportional to the duration of y (or n, etc.); *i.e.*, T seconds. Let $T = M\,\Delta t$. Then these matrices have M elements. What about the dimensions of $\mathbf{H}$? Clearly, the number of columns in $\mathbf{H}$ must be equal to the number of rows in $\mathbf{y}$. Furthermore, we want to have $\mathbf{z}_e$ of length equal to the length of $\mathbf{z}$. Therefore $\mathbf{H}$ must also have at least as many rows as $\mathbf{z}$ (or $\mathbf{y}$). We say "at least" because, clearly, we could assume $\mathbf{H}$ to have more rows than this, but all outputs from $\mathbf{H}$ after $\mathbf{T}$ seconds would be neglected. Thus all of these additional rows of $\mathbf{H}$ may be quite arbitrary. This brings out an important point. We want $\mathbf{H}$ to be an optimum estimator for only the first T seconds of its operation; it can be arbitrary at other times. This fact gives us some freedom when it actually comes to

[6] R. Price, "Optimum detection of random signals in noise, with applications to scatter-multipath communication, I," IRE Trans. on Information Theory, vol. IT-2, pp. 125–135; December, 1956.

[7] See current issues of the *Quart. Prog. Rept.*, Res. Lab. of Electronics, M. I. T., Cambridge, Mass.

building **H**. Having said this much, we can now tacitly assume for our purpose here—namely, determining the minimum variance estimator—that **H** is a square matrix with M rows and M columns.

Therefore, now

$$\epsilon_i = z_i - \sum_{j=0}^{M} h_{ij} y_j ,$$

$$\overline{\epsilon_i^2} = \overline{z_i^2} - 2 \sum_{j=0}^{n} h_{ij} \overline{z_i y_j} + \sum_{j=0}^{M} \sum_{k=0}^{M} h_{ij} h_{ik} \overline{y_j y_k} \tag{30}$$

$$= \overline{z_i^2} - 2 \sum_{j=0}^{M} h_{ij} \phi_{zv}(i, j) + \sum_{j}^{M} \sum_{k}^{M} h_{ij} h_{ik} \phi_{vv}(j, k).$$

For a minimum,

$$\frac{\partial \overline{\epsilon_i^2}}{\partial h_{ij}} = 0 \qquad j, k = 0, 1, \cdots M.$$

Therefore the ith row of **H** must satisfy

$$\phi_{zv}(i, j) = \sum_{k=0}^{M} h_{ik} \phi_{vv}(j, k) \qquad j, k = 0, 1, \cdots M \tag{31}$$

or, rearranging all of the terms in a matrix equation,

$$\mathbf{\Phi}_{zv} = \mathbf{H}\mathbf{\Phi}_{vv} = \mathbf{\Phi}_{zz}$$

since **z** and **n** are statistically independent. Therefore,

$$\mathbf{H} = \mathbf{\Phi}_{zz}\mathbf{\Phi}_{vv}^{-1}$$
$$= (\mathbf{\Phi}_{vv} - \mathbf{\Phi}_{nn})\mathbf{\Phi}_{vv}^{-1} = \mathbf{I} - \mathbf{\Phi}_{nn}\mathbf{\Phi}_{vv}^{-1} .$$

Multiplying both sides of the equation on the left by $\mathbf{\Phi}_{nn}^{-1}$ gives

$$\mathbf{\Phi}_{vv}^{-1} = \mathbf{\Phi}_{nn}^{-1} - \mathbf{\Phi}_{nn}^{-1}\mathbf{H} \tag{32}$$

which is the relation used in (13). The proof that the **H** so obtained actually minimizes $\overline{\epsilon^2}$ is omitted here. (Our method is a direct analog of one given by Levinson.[8] See also Friedland.[9])

Notice from (32) that $\mathbf{\Phi}_{nn}^{-1}\mathbf{H}$, since it is the difference of two symmetrical matrices, is itself symmetric. But what about **H** itself? We see that **H** will be symmetric if $\mathbf{\Phi}_{nn}\mathbf{\Phi}_{vv}^{-1}$ is also symmetric. But even though $\mathbf{\Phi}_{nn}$ and $\mathbf{\Phi}_{vv}^{-1}$ are individually symmetric, their product is not symmetric unless they commute. In general, they do not commute, and therefore **H** is not symmetric. However, if the additive noise is white, $\mathbf{\Phi}_{nn}$ is a scalar matrix and thus commutes with any $\mathbf{\Phi}_{vv}^{-1}$, thereby making **H** always symmetric.

[8] N. Levinson, "The Wiener rms (root mean square) error criterion in filter design and prediction," *J. Math. Phys.*, vol. 25, pp. 261–278; January, 1947.

[9] B. Friedland, "Theory of Time-Varying Sampled-Data Systems," Electronics Res. Lab., Columbia University, New York, N. Y., Tech. Rept. T-19/B; April, 1957.

Finally, consider the identity

$$\begin{bmatrix} x_0 & x_1 \end{bmatrix} \begin{bmatrix} a & c \\ c & b \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = ax_0^2 + 2c\, x_0\, x_1 + bx_1^2$$

$$= \begin{bmatrix} x_0 & x_1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 2c & b \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}.$$

This indicates that if in the matrix of a symmetric quadratic form, we remove all terms above the main diagonal and double all terms below it, the value of the quadratic form is unchanged. However, we now have a matrix that can represent a realizable filter. This is the relation used in (17).

Appendix II

Proof that W of (19) is a "Whitening" Filter

We shall first derive a formula for the covariance matrix of the output $z(t)$ of a filter W, for a random input $n(t)$. The input-output relation for this situation is [*cf.* (3)]

$$\mathbf{z} = \mathbf{W}\mathbf{n}. \tag{33}$$

The covariance matrix of **z** can be found by taking the direct product (or Kronecker product) of **z** and $\mathbf{z}_t$. Thus

$$\mathbf{\Phi}_{zz} = \overline{\mathbf{z} \otimes \mathbf{z}_t} \qquad \begin{bmatrix} \text{the bar denotes an ensemble} \\ \text{average over the inputs } x(t) \end{bmatrix}$$
$$= \mathbf{W}\overline{\mathbf{n} \otimes \mathbf{n}_t}\mathbf{W}_t$$
$$= \mathbf{W}\mathbf{\Phi}_{nn}\mathbf{W}_t . \tag{34}$$

Now $\mathbf{\Phi}_{nn}$ is a covariance matrix and is, therefore, positive definite and can hence be written,

$$\mathbf{\Phi}_{nn} = \mathbf{B}\mathbf{B}_t \tag{35}$$

where **B** is a triangular matrix; that is, **B** has all zeros above the main diagonal. Then

$$\mathbf{\Phi}_{zz} = \mathbf{W}\mathbf{B}\mathbf{B}_t\mathbf{W}_t . \tag{36}$$

If we wish $z(t)$ to be "white," we must have $\mathbf{\Phi}_{zz} = \mathbf{I}$. This can be obtained by selecting filter **W** so that

$$\mathbf{W} = \mathbf{B}^{-1}. \tag{37}$$

Then **W** is a "whitening" filter. Now note that

$$\mathbf{W}_t\mathbf{W} = \mathbf{B}_t^{-1}\mathbf{B}^{-1} = (\mathbf{B}\mathbf{B}_t)^{-1} = \mathbf{\Phi}_{nn}^{-1} \tag{38}$$

which is (19) of the text.

VI. Acknowledgment

This particular problem was originally suggested to the author by Dr. R. Price of Lincoln Laboratory, M. I. T., Cambridge, Mass. The study of Price's pioneering work in this field[6] and helpful conversations with him and Prof. J. M. Wozencraft of M. I. T. were the chief spark for this work.

A Matched Filter Communication System for Multipath Channels*

STEVEN M. SUSSMAN†

Summary—**A matched filter communication system is described whose underlying principles are based on the Rake. The point-to-point synchronous teletype system employs complex lumped-parameter networks to generate and receive a pair of long-duration, broadband signals representing Mark and Space respectively. The receiver contains a pair of matched filters whose output is a narrow pulse when the matching waveform is applied. One advantage of the system, arising from the long duration of the signals, is an increase in energy per teletype baud when operating under a peak power limitation. Another is that multiple propagation paths due to ionospheric reflection are resolved by the broadband signals, resulting in the appearance of the multipath pattern at the output of the Mark or Space matched filter. Recombination of paths is achieved by means of a recirculating delay line tuned to the teletype baud rate in conjunction with parallel multiplier-integrators in the Mark and Space channels. The combination acts as a self-adjusting correlation detector for the multipath pattern.**

I. Introduction

THE SUBJECT of this paper is the theoretical aspects of a recent development in radio-teletype communication via the ionosphere. The ionospheric channels are characterized by multiple propagation paths, resulting in echoes for pulse transmission and in selective fading for narrowband waveforms. The fading or interference phenomenon is due to the superposition of signals arriving at the antenna with different phase-shifts after passage over transmission paths of different lengths. The shortcomings of narrowband signals under multipath conditions have impelled development of systems employing broadband signals capable of resolving the path structure. Once the paths have been resolved the signals can be combined algebraically rather than vectorially as in the narrowband case. One such wideband system is the "Rake"[1] whose basic principles also underlie the present system.

In radio-teletype communication the fundamental information-carrying symbols are the "Mark" and "Space" represented in transmission by characteristic signals referred to as "bauds". Frequency-shift keying (FSK) systems, for example, represent Mark and Space by two different tones. In principle, the large bandwidth requirement stated in the previous paragraph could be satisfied by transmitting short pulses spaced sufficiently far apart to avoid intersymbol interference. However, practical restrictions on transmitter peak power would severely limit the energy per teletype baud. Much higher efficiencies in this respect are achieved by operating the transmitter continuously. One is thus led to a qualitative specification on the transmitted signals to be broadband (for resolution) and of long duration (for energy) *i.e.*, to have large time-bandwidth (TW) products. We have tacitly assumed binary operation, similar to FSK, in representing Mark and Space by signals of the specified form. The representation could also have been made for blocks of symbols, but this would involve considerable increase in complexity in both the analysis and the implementation.

For the binary case we can draw upon the theory of detection of signals in noise to find the receiver which minimizes error rate.[2-5] Since the signals are to be detected using passive networks as matched filters, we can, as a natural consequence, generate the signals by impulsively exciting similar passive networks. In the absence of multipath the basic communication system has the form shown in Fig. 1. When a Mark is to be sent, the Mark generating filter is excited by an impulse, and the corresponding impulse response, which has the desired large TW product, is transmitted. At the receiver a pair of filters matched to the Mark and Space waveforms respectively perform the detection. The impulse response of the Mark matched filter is the time-reverse (delayed) of the Mark waveform so that the output as expressed via the convolution integral is the auto-correlation function of the Mark signal. Because of the large TW product

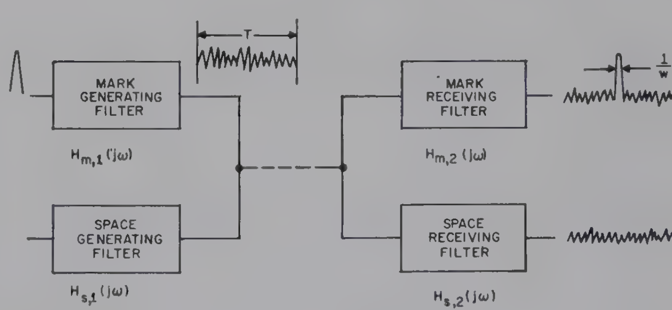

Fig. 1—Basic matched filter communication system.

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† Melpar, Inc., Falls Church, Va.

[1] R. Price and P. E. Green, Jr., "A communication technique for multipath channels," Proc. IRE, vol. 46, pp. 555–570; March, 1958.

[2] P. M. Woodward and I. L. Davies, "Information theory and Inverse probability in telecommunication," *Proc. IEE*, vol. 99, pt. 3; March, 1952.

[3] C. W. Helstrom, "The resolution of signals in white Gaussian noise," Proc. IRE, vol. 43, pp. 111–1118; September, 1955.

[4] G. L. Turin, "Communication through noisy, random-multipath channels," 1956 IRE Convention Record, pt. 4, pp. 154–166.

[5] D. Middleton and D. Van Meter, "Detection and extraction of signals in noise from the point of view statistical decision theory," *J. SIAM*, vol. 3, December, 1955; vol. 4, June, 1955.

of the signal the autocorrelation function will have a sharp central spike with low-level contributions extending a time T in either direction. Concurrently, the Space matched filter displays the cross-correlation of its impulse response with the received signal, *i.e.*, the cross-correlation between the Mark and Space waveforms. For reasons to be clarified shortly the waveforms for Mark and Space are chosen to make the Mark-Space cross-correlation as small as possible relative to the peak of the auto-correlation function. The decision between Mark and Space is based on the sign of the difference between the two matched filter outputs sampled at the instant of the peak of the auto-correlation function. This procedure is optimum with respect to error rate for equi-probable, equi-energy signals in white Gaussian noise.[3,4]

Unfortunately ionospheric propagation phenomena do not permit the implementation of the simple binary matched-filter communication system as sketched above. Multiple propagation paths result in multiple correlation pulses at the output of the matched filter. The pulses will fluctuate in amplitude and delay and will occasionally drop out to reappear at a different delay. Under these conditions sampling of a correlation peak, say the one corresponding to the strongest path, is extremely difficult (besides being nonoptimum, since the energy of the remaining paths is ignored). To cope with this problem, the Rake technique for multipath channels was developed. In essence, the existing multipath pattern is measured by averaging the signal strength over the recent past for a large number of delay increments. The measured pattern is then used as a local reference in a correlation detection. For a more detailed discussion of the Rake idea and its theoretical foundation we refer to Price and Green.[1] The system described in the following section is an implementation based on matched filters rather than on correlation by multiplying and integrating. The utilization of matched filters yields a number of important benefits as outlined in the conclusion.

II. Multipath Combining Technique

In Fig. 2 we trace the signal from the output of the matched filters, consideration of whose realization is deferred to Section III. The multipath pattern appears as a sequence of pulses at the output of the Mark or Space filter depending on which symbol was transmitted. The outputs are summed thus assuring that the multipath pattern, for the moment assumed invariant, is available for each baud at the input of the delay line. Advantage is now taken of the periodicity of the pattern resulting from the fixed baud period T'. The delay-line and positive feedback (less than unity) form a recirculating storage to which is added the most recent matched filter output. The period of the pulse pattern being equal to the delay insures that successive patterns add in phase, thereby building up the stored signal in accordance with the series.

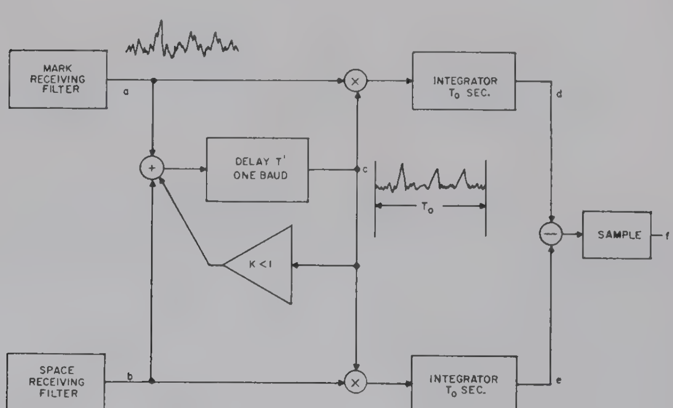

Fig. 2—Multipath combining circuitry.

$$1 + K + K^2 \cdots = \frac{1}{1 - K}. \tag{1}$$

Additive noise which may be mixed with the signal adds incoherently or power-wise since noise samples separated by T' seconds will be independent. The noise power in the loop grows as

$$1 + K^2 + K^4 + \cdots = \frac{1}{1 - K^2}. \tag{2}$$

The improvement in signal-to-noise power ratio due to the feedback loop is

$$\left(\frac{1}{1 - K}\right)^2 \Big/ \frac{1}{1 - K^2} = \frac{1 + K}{1 - K}. \tag{3}$$

The output of the delay-line may be regarded as a "cleaned-up" estimate of the multipath pattern which is now to be used as the local reference in a correlation detection for the presence of the pulse pattern at either of the two matched filter outputs. After multiplying and integrating in each channel for a time T_0 sufficient to cover the multipath spread, the Mark and Space integrals are compared and a decision made in favor of the larger one.

A system evaluation can be based on the signal-to-noise ratio $(S/N)_{\text{out}}$ at the decision point f. For Gaussian noise statistics $(S/N)_{\text{out}}$ is then easily converted to error rate. Appendix I contains a derivation of $(S/N)_{\text{out}}$, under some simplifying assumptions which are stated there, for rectangular signal and noise spectra extending from 0 to W cycles per second. The result is

$$\left(\frac{S}{N}\right)_{\text{out}} = \frac{\dfrac{2E}{N_0}}{\dfrac{8WT_0}{2E/N_0}\left(\dfrac{1 - K}{1 + K}\right) + \dfrac{4}{1 + K}} \tag{4}$$

where E is the energy per baud and N_0 the noise power spectral density. This relation expresses the influence on system performance of the parameters in the idealized configuration of Fig. 2. The optimum behaviour $(S/N)_{\text{out}} = E/N_0$ is attained when $K = 1$. In practice K is less than one to prevent oscillation of the loop and also to accommodate slowly varying multipath patterns.

he role of the integration time T_0 is seen to depend rongly on $2E/N_0\ (1 + K)/(1 - K)$. For reasonably rge $2E/N_0$ (the signal-to-noise ratio at the output of the atched filter) and K not small, the choice of T_0 is not ery critical provided $T_0W \gg 1$ to satisfy the assumptions the derivation of (4). This means that the receiver ming, whose chief function is to establish the end points the interval T_0, need not be very precise so long as T_0 raddles the multipath pattern.

Up to now the system's RF portions which are interosed between the generating filter and receiver have een ignored. Since the physical constraints on the filter alization restrict operation to the audio-frequency nge, modulation and demodulation are required. To reserve linear operations on the received signal, the ceiver employs frequency-translation to dc. We deberately avoid the term synchronous demodulation nce phase coherence is not required, as can be seen from e following. Assume for the moment no frequency error ut a constant phase-shift introduced in transmission ther through ionospheric reflection or phase-shift etween transmitter and receiver oscillators. For doubledeband suppressed carrier transmission a phase-shift applies a multiplicative factor $\cos\theta$ to the demodulated gnal and this factor effectively reduces the signal energy. or single sideband, on the other hand, the effects are ore subtle. The phase-shift is translated to low frequency aving the signal spectrum at the output of the matched lter in the form

$$S(\omega) = \begin{cases} Ge^{-i\theta} & 0 < \omega \leq 2\pi W \\ Ge^{+i\theta} & 0 > \omega \geq -2\pi W \end{cases}. \tag{5}$$

he normalized output is the Fourier transform of this ectrum, giving

$$\phi(\tau, \theta) = \cos\theta \frac{\sin 2\pi W\tau}{2\pi W\tau} + \sin\theta \frac{1 - \cos 2\pi W\tau}{2\pi W\tau} \tag{6}$$

hich reduces to the former case when $\theta = 0$. The relevant roperty of this function is that

$$\int_{-\infty}^{\infty} \phi^2(\tau, \theta)\, d\tau = \frac{1}{2W} \tag{7}$$

dependent of the phase shift θ.[6] The implications of this ct are that for reasonably stable phase conditions the rculating loop (Fig. 2) stores $\phi(\tau, \theta)$; multiplication with e received signal and integration forms (7) approxiately, and the result is independent of θ. Thus for SSB odulation the signal suffers virtually no degradation due arbitrary phase-shift in the channel. This feature is tributable to the operation of the feedback loop and rrelation circuitry which performs somewhat like a lf-tuning optimum detector for the matched filter itputs.

The phase stability required to achieve the above effect is related to the memory time of the loop. The stored waveform is the superposition of past phase patterns of the form $\phi(\tau, \theta)$, and if the phase structure changes appreciably during the memory span of the loop, the stored pattern will not resemble the most recent signal and a coherent build-up will not occur. Hence the rate of change of phase (frequency error) must be held small relative to the inverse time-constant of the loop. But the time-constant is adjustable through K and can therefore be adapted to prevailing conditions. Such an adjustment will have the desired outcome only as long as the phase rate remains small enough to have negligible effect on the matched filter output itself.

III. Signal Generation

The generating and matched filter realization is initiated by again considering Fig. 1. The matched condition in the frequency domain states that the generating and receiving filter transfer functions be conjugate, except for an arbitrary linear phase included for realizability, *i.e.*

$$H_{m,2}(j\omega) = H_{m,1}(-j\omega)e^{-i\omega T} \tag{8}$$

whence

$$H_{m,1}(j\omega)H_{m,2}(j\omega) = |H_{m,1}(j\omega)|^2\, e^{-i\omega T}. \tag{9}$$

We are at liberty to choose the magnitude $|H_{m,1}(j\omega)|$ to be constant over the available bandwidth and zero outside.[7] This choice is satisfying in that the band is fully utilized to achieve a broadband signal, although there may be some improvement to be gained by shaping the band to attain a desirable pulse shape at the matched filter output. Ignoring the last point, the cascade of generating filter and matched filter appears as a pure delay-line over the band of interest. The delay-line is split between transmitter and receiver so as to cause the generating filter to have the desired broadband, long-duration impulse response. It follows that $H_{m,1}(j\omega)$ must have a nonconstant delay vs frequency curve, for then different portions of the spectrum of the excitation impulse will be delayed by different amounts and the response will be a stretched-out waveform.

A particularly convenient realization for both generating and matched filter is in the form of a cascade of all-pass, constant-resistance bridged-T networks. This elegant solution to the design problem is due to E. A. Guillemin of Massachusetts Institute of Technology.

The pole-zero configuration for the over-all cascade is shown in Fig. 3. The symmetrical placement of zeroes and poles about the $j\omega$-axis characterizes an all-pass structure, while the uniform spacing of singularities parallel to the axis yields a nearly linear increase of phase with frequency along the $j\omega$-axis. Since the delay imposed on any frequency group (spectral components in the vicinity of a

[6] It can be shown, using the Hilbert transform representation r the SSB signals, that this property is not restricted to rectangular ectra but is true for any signal spectrum.

[7] It is of interest to note that if $|H_{m,1}(j\omega)|$ is constant, and only then, the conjugate network coincides with the reciprocal network except for flat gain.

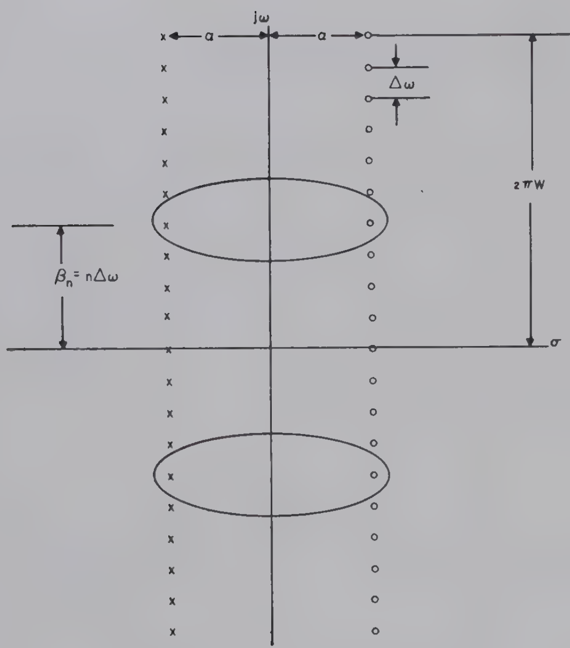

Fig. 3—Pole zero parameters.

particular frequency) equals the derivative of phase with respect to frequency at the corresponding point, the configuration represents an all-pass, constant delay.

From this pole-zero pattern we obtain both the generating and matched filter by assigning singularities, which are realized in groups of four as circled in Fig. 3, in a more or less random fashion to one or the other filter. The generating filter will consist of a portion of the pole-zero pairs, and its phase-slope or delay will vary with frequency in a manner determined by the particular assignment of singularities. The matched filter contains those pole-zero pairs of the total array which are not present in the generating filter. Since each filter is individually all-pass, the two form a conjugate pair, except for the constant delay, as required by the matching condition (8). The design details and network analysis are contained in Appendix II.

The distribution of pole-zero pairs between the two filters is largely arbitrary. As developed in Appendix II, the number of pole-zero pairs is of the order of $2TW$, a large number in our application. Of the possible pole-zero assignments a great many will be acceptable in that they yield long duration impulse responses. Certain others are obviously undesirable and should be avoided, *e.g.* assignment of alternate pairs to each filter which results in each filter being approximately a delay-line of half the total delay. To insure that a substantial amount of transmitted signal energy appears at the extremes of the baud, it is necessary to provide spectral regions of long delay and of short delay in the transmitter filter. This is achieved by assigning long strings of adjacent pole-zeros at both terminals thereby introducing a large delay at the corresponding frequencies.

As seen in Appendix II, large delay is associated with high attenuation due to the inevitable losses in the network elements. Consequently, the envelope of the impulse response will show an exponential decay in addition to other fluctuations resulting from the particular pole-zero assignment. Extreme nonuniformity of the signal envelope tends to defeat one aim of the system which is efficient transmitter operation under peak power restrictions. To counteract this tendency, the signal may be limited in amplitude prior to transmission. It is expected that the increase in transmitted energy per baud made possible by limiting will more than counterbalance the loss in signal-to-noise ratio due to the nonlinearity.

In the preceding, the baud repetition period T' has been distinguished from the delay T of the cascade networks. The choice of these two parameters is influenced by the range of multipath delays and the possibility of intersymbol interference. As pointed out earlier, the response of the matched filter extends approximately for a time T on either side of the correlation peak. Interference arises when the correlation function tail of one baud overlaps the main pulse or pulses of the succeeding symbol. The effect of the overlap is to introduce an additional noise term proportional to the signal level in the tail. If T' exceeds T by more than the difference between the longest and the shortest path delay, intersymbol interference will not occur. Under these conditions, and also when $T' = T'$ gaps will appear in the transmitted waveform since the signal energy per baud does not occupy the full interval T'.

One further point is worth noting here. Because the complete cascade of bridged-T sections has all the properties of a delay line, it can well serve as the delay in the multipath tracking loop. This step will avoid a second difficult network synthesis problem since delay lines having large delay-bandwidth products are not trivial to design.

The discussion of filter synthesis so far has focussed on the design of either the Mark of the Space generating and matched filter pair. We now consider a specific relation between the Mark and Space pole-zero assignment which insures that the unmatched filter output is low compared to the correlation peak from the matched filter. This is achieved by making the Mark generating filter identical to the Space matched filter, and similarly for the other two. Thus, in Fig. 1 the signal path along the diagonals (through $H_{m,1}$ and $H_{s,2}$ say) passes twice through the same network, whereby the phase distortion is doubled, and the stretch-out effect is twice that of each filter alone. The unmatched filter output will therefore be the transmitted signal on an expanded time scale and consequently no strong peaks can build up as in the case of the conjugate or matched filter. If the impulse response of the generating filter has a fairly uniform energy distribution over the baud length T, so will the unmatched filter output. It follows that, since the integration time T_0 in the multipath combiner is very likely a small fraction of T, we may justify the neglect of the contribution from the unmatched filter in the appended analysis especially when the signal-to-noise ratio is low.

IV. Conclusion

The utilization of matched filters in conjunction with novel multipath combining scheme promises a number of nportant advantages over the earlier Rake technique ased on a multiplicity of correlators. In the matched lter system the major portion of the signal processing is erformed by passive networks. This leads to a smaller ver-all package as well as increased reliability and ease f maintenance.

In some respects the matched filter system is highly exible: various multipath fluctuation rates can be acommodated by simply changing the feedback gain K, ne integration time T_0 and its epoch can also be adjusted ccording to the existing multipath pattern. On the other and, the filters themselves are not easily variable; hence ne delay T is fixed and the baud period T' is variable nly over a small range.

Finally we note that the matched filter synthesis method eported herein is of importance in its own right, having pplicability to pulse-compression radar and similar ystems, as well as to communication through multipath hannels.

Appendix I
Multipath Combiner Analysis

We consider a rectangular passband $0 - W$ per cycles er second. The corresponding normalized correlation unction for both signal and noise is

$$\phi(\tau) = \frac{\sin 2\pi W\tau}{2\pi W\tau}. \tag{10}$$

ssuming a Mark is sent, the signal output of the Mark lter has the form[8]

$$s_a(t) = \sqrt{2EW} \sum_i a_i \phi(t - t_i). \tag{11}$$

Iere E is the total energy per baud; the a_i are path mplitudes and t_i the relative path delays. The path tructure is assumed to be fixed and the paths sufficiently eparated so that

$$\phi(t_i - t_j) \approx 0 \qquad i \neq j. \tag{12}$$

ince the total energy per baud is $E = \int s_a^2(t)dt$, it follows hat

$$\sum_i a_i^2 = 1. \tag{13}$$

'he noise output from the Mark filter is $n_a(t)$ with mean ero, power $\overline{n_a^2} = N$ and correlation function $R_n(\tau) =$ $V\phi(\tau)$. Similarly at the Space filter for $n_b(t)$ which is ndependent of $n_a(t)$ when the filters are reasonably omplex. Further, the pair of filters are assumed to be well esigned so that the output of the Space filter when a Iark is sent is sufficiently low for all time to be ignored elative to $n_b(t)$. This assumption is justified for small signal-to-noise ratios under which the system is expected to operate.

For fixed multipath conditions the loop signal output is

$$s_c(t) = \sqrt{2EW} \frac{1}{1 - K} \sum_i a_i \phi(t - t_i). \tag{14}$$

The noise at c has mean zero, power $\overline{n_c^2} = 2N/(1 - K^2)$ and is independent of n_a and n_b due to the delay T.

Multiplying and integrating gives for the outputs at points d and e, respectively

$$E_d = \int_{-T_0/2}^{+T_0/2} [s_c(t) + n_c(t)][s_a(t) + n_a(t)]\, dt \tag{15}$$

$$E_e = \int_{-T_0/2}^{T_0/2} [s_c(t) + n_c(t)] n_b(t)\, dt. \tag{16}$$

At the decision point f we have

$$\left(\frac{S}{N}\right)_{\text{out}} = \frac{(\overline{E_d} - \overline{E_e})^2}{\overline{(E_d - E_e)^2} - \overline{(E_d - E_e)}^2} = \frac{\overline{E_d}^2}{\overline{E_d^2} + \overline{E_e^2} - \overline{E_d}^2} \tag{17}$$

where we have used the fact that $\overline{E_e} = 0$ and $\overline{E_e E_d} =$ $\overline{E_e}\,\overline{E_d}$ due to the independence of the noise terms, averages being taken over the noise ensemble.

We now proceed to evaluate $\overline{E_d}$, $\overline{E_d^2}$ and $\overline{E_e^2}$:

$$\overline{E_d} = \int_{-T_0/2}^{T_0/2} s_c(t) s_a(t)\, dt$$

$$= \frac{2EW}{1 - K} \int_{-T_0/2}^{T_0/2} \sum_i \sum_j a_i a_j \phi(t - t_i)\phi(t - t_j)\, dt. \tag{18}$$

The integration time T_0 will encompass the entire multiple pulse pattern and $T_0 W \gg 1$. Hence as far as the signal contribution is concerned we may let the limits of integration go to infinity. Then making use of the reproductive property of $\phi(t)$ under convolution we obtain via (12) and (13)

$$\overline{E_d} = \frac{E}{1 - K}. \tag{19}$$

Turning to $\overline{E_d^2}$ we have by writing the square of (15) as a double integral and averaging under the integral

$$\begin{aligned}\overline{E_d^2} = \int_{-T_0/2}^{T_0/2} \int_{-T_0/2}^{T_0/2} \{ & s_c(t_1)s_c(t_2)s_a(t_1)s_a(t_2) \\ & + \overline{n_c(t_1)n_c(t_2)}\; \overline{n_a(t_1)n_a(t_2)} \\ & + s_c(t_1)s_c(t_2)\overline{n_a(t_1)n_a(t_2)} + s_a(t_1)s_a(t_2)\overline{n_c(t_1)n_c(t_2)} \\ & + \text{cross terms which average to zero} \} dt_1 dt_2.\end{aligned} \tag{20}$$

The first term in (20) is simply $\overline{E_d}^2$. The second term is

$$\frac{2N^2}{1 - K^2} \int_{-T_0/2}^{T_0/2} \int_{-T_0/2}^{T_0/2} \phi^2(t_1 - t_2)\, dt_1\, dt_2 .$$

Since $\phi(\tau)$ is an even function this reduces via a simple change of variables to

$$\frac{2N^2}{1 - K^2} \int_{-T_0}^{T_0} (T_0 - |\tau|)\, \phi^2(\tau)\, d\tau.$$

[8] Supscripts refer to corresponding points in Fig. 2.

For $T_0 W \gg 1$, $\phi^2(\tau)$ decays rapidly and we can ignore $|\tau|$ relative to T_0 and again let the limits go to infinity. The second term in (20) then becomes

$$\frac{N^2}{(1-K^2)}\frac{T_0}{W}.$$

The last two terms are

$$\left[\frac{2EWN}{(1-K)^2} + \frac{4EWN}{1-K^2}\right]$$

$$\cdot \int_{-T_0/2}^{+T_0/2} \int_{-T_0/2}^{+T_0/2} \phi(t_1)\phi(t_2)\phi(t_1 - t_2)\, dt_1\, dt_2$$

$$= \frac{EN}{2W(1-K)^2} + \frac{EN}{W(1-K)^2}$$

where the previous approximations and convolution property of $\phi(\tau)$ have been applied. This completes the evaluation of $\overline{E_a^2}$. Similarly

$$\overline{E_e^2} = \frac{EN}{2W(1-K)^2} + \frac{N^2 T_0}{(1-K^2)W}.$$

Finally, setting $N/W = N_0$, inserting in (17), and simplifying gives (4).

Appendix II
Synthesis of the Desired Networks

The synthesis of a transfer function having a symmetrically placed pair of poles and zeros, *e.g.* those circled in Fig. 3, follows well known procedures. Using the constant-resistance lattice terminated in a 1-ohm resistance as the basic structure [see Fig. 4(a)] we have

$$Z_a Z_b = 1$$

$$\frac{E_2}{E_1} = \frac{E_2}{I_1} = Z_{12} = \frac{1 - Z_a}{1 + Z_a}$$

$$Z_a = \frac{1 - Z_{12}}{1 + Z_{12}}.$$

The desired transfer function is

$$Z_{12} = \frac{(s-\alpha)^2 + \beta_n^2}{(s+\alpha)^2 + \beta_n^2}$$

whence

$$Z_a = \frac{1}{\dfrac{s}{2\alpha} + \dfrac{\alpha^2 + \beta_n^2}{2\alpha}\dfrac{1}{s}}$$

and Z_b is its dual. The lattice has the elements shown in Fig. 4(b) and can be unbalanced to yield the bridged-T of Fig. 4(c). Note that realization in this form is possible only if $\beta^2 > 3\alpha^2$; for zeroes closer to the real axis a different procedure must be used.[9]

The bridged-T networks corresponding to any combination of conjugate pole-zero pairs can be connected

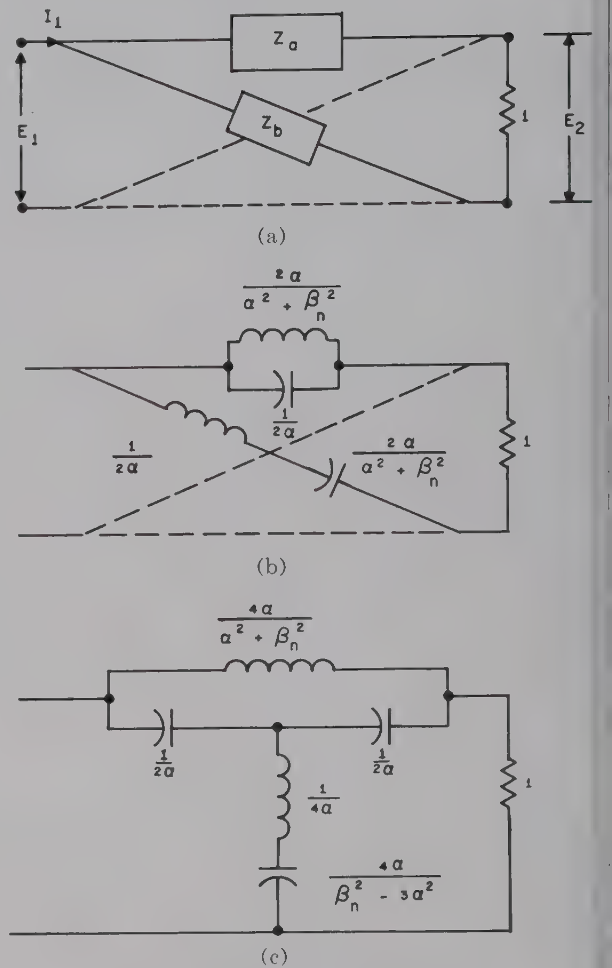

Fig. 4—Network configurations.

directly in cascade, the constant-resistance input impedance of one forming the load for the preceding one.

Before proceeding to an analysis of the cascade networks we introduce the effect of incidental dissipation in the coils. This produces a shift of the entire pole-zero pattern to the left by an amount δ. We assume that the necessary adjustments in component losses have been made so that all singularities are shifted by the same amount. The transfer function for the entire array now reads

$$H(s) = \sum_{n=-N}^{N} \frac{(s - \alpha + \delta)^2 + \beta_n^2}{(s + \alpha + \delta)^2 + \beta_n^2} \qquad \beta_n = n\Delta\omega$$

which is no longer an all-pass network.

An excellent treatment of this network function is given by Guillemin,[9] and we repeat here only the pertinent results. For a long array the delay is very nearly $T = 2\pi/\Delta\omega$, independent of the parameter α, except for fringing at the ends of the array. However, the fringing effect can be largely eliminated by adding a few judiciously placed pole-zero pairs near the edges of the frequency band. The ripple in delay due to the discrete nature of the singularities can be made negligible by choosing α several times greater than $\Delta\omega$. To cover a range of W cycles per second with a delay T, singularities being spaced by $\Delta\omega = 2\pi/T$ radians, the cascade network must contain

[9] E. A. Guillemin, "Synthesis of Passive Networks," John Wiley and Sons, Inc., New York, N. Y.; pp. 493, 639 ff; 1957.

'T of the bridged-T sections, plus whatever fringing-ompensation sections are necessary.

The incidental dissipation manifested by the shift δ, ıtroduces frequency dependence into the magnitude ınction of each section, *viz.*

$$| H_n(j\omega) |^2 = \frac{(\alpha - \delta)^2 + (\omega - \beta_n)^2}{(\alpha + \delta)^2 + (\omega - \beta_n)^2}$$

'hich reaches a minimum at the section frequency β_n. 'he attenuation affects the system performance adversely ı that the generating and matched filter functions are o longer strictly conjugate since their magnitudes are ot identical. This follows from the fact that the attenua-ion of the nth bridged-T section applies at the transmitter r the receiver but not at both. The loss due to the mis-ıatch will depend on the pole-zero assignment, but will e unimportant for reasonably high-Q coils. A second ractical consequence of incidental dissipation is the omposite attenuation of the entire cascade of sections. 'he magnitude of the finite array is essentially uniform within the useful band of the filters, and is given approximately by

$$| H(j\omega) | = e^{-2\pi\delta/\Delta\omega}.$$

Outside the band the magnitude increases and approaches asymptotically to unity, *i.e.*, the shifted array tends to behave like a bandstop filter rather than a bandpass filter. To overcome this undesirable behavior the system must include a bandpass filter restricting the signal and noise energy to the spectral region where the filters are actually matched.

Acknowledgment

In addition to the previously noted valuable assist from E. A. Guillemin, credit for the work reported herein is due to the Melpar project group Melpar, Inc., Falls Church, Va., and its technical supervisors, in particular Drs. T. P. Cheatham, Jr., D. Van Meter and A. Kohlenberg.

A Matched Filter Detection System for Complicated Doppler Shifted Signals*

ROBERT M. LERNER†

Summary—A matched filter system is described which was esigned to detect complicated signals subject to a wide range of ossible Doppler shifts. A 100 tap band-pass delay line was used in onjunction with a resistor weighting matrix to synthesize signals nd filter characteristics. The system could handle a signal with duration-bandwidth product of 100 over a range of Doppler equency shifts 17 times the reciprocal of the signal duration. A ıeoretical discussion of the Doppler effect is given, making use conjugate functions or Hilbert Transforms. Various engineering ompromises which simplify the construction of matched filters e suggested. The performance of the resulting signal detection stem was within 5 db of that of an ideal theoretical model.

I. Introduction

As THE ART of designing signals becomes better understood, more and more emphasis is being placed on the design and use of waveforms for 'hich the product of waveform duration T and band-'idth W is a number much greater than one. Some TW parameters can be independently specified in constructing such a waveform[1] so that its most general form resembles a sample of random noise having the given duration and bandwidth. These "complicated" signal waveforms are useful whenever there is an apparent conflict between the desire to deliver signal energy slowly and the desire for the high time resolution inherent in the use of a broad bandwidth. Several authors [1], [3], [5], [6] have suggested and discussed the use of such signals in the design of radar systems in which it is desired to make accurate simultaneous measurements of velocity (implying long pulse duration) and range (implying wide signal bandwidth) of an object. Price and Green [12], and also Sussman [14], used such signals to measure and combat the multipath structure of the ionosphere in HF transmission. Darlington [16] has used them to overcome peak-power limitations in wide-band power amplifiers. Such signal waveforms have also been used in combating impulsive disturbances [15] in communication systems.

The use of "complicated" signals requires a degree of sophistication in signal design and detection that is often

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[1] See C. E. Shannon, "Communication in the presence of noise," Proc. IRE, vol. 37, pp. 10–21; January, 1949; also D. Gabor [2].

not necessary with "simpler" waveforms (*i.e.*, duration-bandwidth product about equal to one[2]). Let us assume a signaling system in which the transmitter selects and sends out one of a group of possible waveforms. The transmission medium will produce changes in the transmitted waveform, some of them random, so that the receiver must guess which signal, if any, was transmitted. It is by now well established [1]–[3] that almost any statistically "optimum" procedure for distinguishing amongst a group of signals (including of course the absence of signal) masked by additive white Gaussian noise is equivalent to a coherent detection in which the integrals I_k

$$I_k = \int_{-\infty}^{\infty} y(t) s_k(t)\, dt \tag{1}$$

are compared with each other and with certain thresholds. In these integrals, $y(t)$ is the signal actually received and the $s_k(t)$ are the various signal waveforms to be expected in the absence of noise. If the integral (1) is computed in actual apparatus by multiplication and integration, the detection process is called "correlation detection." If the integral is obtained by examining the output of a linear filter at the appropriate time, the process is referred to as "matched filter detection."[3]

In a communications system, a symbol[4] is transmitted by assigning to it a unique waveform. If the transmission path subjects the signal waveform to a frequency shift due to Doppler effect or unsynchronized single-sideband oscillators, then the receiver must contend not with the unique transmitted waveform but with a whole gamut of waveforms extending over the likely range of the frequency shift. If the Doppler[5] shift is ignored in the detection process, usually no degradation results, provided that the product of the amount of frequency shift and the duration of the signal waveform is much less than unity. However, if the Doppler-duration product is about one or greater, the Doppler shift cannot be ignored. It then becomes necessary to compute integrals, I_k, corresponding to a number of frequency translates of the originally transmitted waveform.[6]

In the fall and winter 1954–55, the author and some of his associates at Lincoln Laboratory undertook the project of demonstrating the feasibility of constructing a bank of filters matched to an arbitrary complicated signal, having a duration-bandwidth product of 100, in which the signal spectrum might be subjected to a wide range of frequency shifts.

Signals and filter responses were constructed at band-pass by adding up the outputs from equally spaced taps on a delay line with weights a_k:

$$f(t) = \sum a_k u(t - k\tau) \tag{2}$$

in which $f(t)$ is the desired response, τ is the tap spacing, and $u(t)$ is the form of the transient response at each tap. $u(t)$ was chosen to be a band-pass transient whose duration was about equal to the tap spacing; the length of the line, T, was so chosen that there were 100 taps. A resistor matrix was used to construct enough orthogonal filter characteristics, matched to Doppler shifts of an arbitrary signal, to permit satisfactory matched filter detection for Doppler-duration products up to 17.

This paper gives an appropriate mathematical description of the effect of Doppler shift on an arbitrary signal, describes the equipment that was constructed to implement the results of this analysis, and gives some of the experimental results of tests made on that equipment.

II. The Representation and Detection of Doppler Shifted Signals

In order to build filters matched to a frequency translates of a signal $f(t)$, it is desirable to have an expression for the signal after its spectrum has been shifted in frequency by Δ radians per second. Such an expression can be obtained from the complex signal introduced by Gabor [10]. Except for a fixed delay, the shifted signal $g(t)$ is given by

$$g(t) = f(t) \cos(\Delta t + \lambda) - \hat{f}(t) \sin(\Delta t + \lambda) \tag{3}$$

in which λ is an arbitrary phase angle and $\hat{f}(t)$ is the "conjugate function" or Hilbert transform (see below) of $f(t)$.

It is usually taken for granted that the result of Doppler effect in a signaling system is to introduce a frequency shift Δ into the signal spectrum. Indeed, the filters to be described later on function only in the event that this supposition is for all practical purposes correct. Accordingly, we wish to discuss the circumstances under which the Doppler effect is equivalent to a frequency shift. A slight extension of this discussion establishes (3) and yields an expression for the output of a filter matched to one frequency translate of $f(t)$ when it is excited by still another frequency translate.

A Doppler shift is an effect of continuously stretching or compressing the path length traversed by a signal during transmission. It appears as an apparent time scale stretch between transmission and reception. Thus if t is the time scale of the transmitter and t_r is that at the receiver,

$$t_r = \alpha t. \tag{4}$$

[2] For a more complete discussion of the duration-bandwidth product of a signal waveform, see Gabor [10] and also R. M. Lerner, "The representation of signals," IRE Trans. on Information Theory, vol. IT-5, pp. 197–216; May, 1959. (Also IRE Trans. on Circuit Theory, vol. CT-6, pp. 197–216; May, 1959.)

[3] See Turin [13] in this issue.

[4] In a radar, reflection from a discrete object takes the place of the "symbol."

[5] Except where the context indicates otherwise, the term "Doppler shift" will be used to refer to any *a priori* unknown displacement of the signal spectrum, whether or not it is due to the Doppler effect.

[6] Clearly, additive Gaussian noise and a frequency shift are not the only ways in which the transmission path could affect a transmitted signal. The consideration of further possibilities such as a dispersive path or non-Gaussian noise is beyond the scope of this paper.

us, if the transmitted signal is $f_t(t)$, then the form of e received signal, except for a constant delay, is given by

$$f_r(t) = f_t(\alpha t). \tag{5}$$

e scale factor α differs from unity by an amount which proportional to the ratio of the rate of change of effective nsmission path length to the velocity of propagation ng that path. Ordinarily the rate of change of path gth is small compared with the propagation velocity, d

$$\alpha = 1 + \delta \tag{6}$$

ere δ' is small compared with one.

The importance of the Doppler effect depends on the ation of signal examined by the receiver, the size of and the highest frequency component in the transmitted nal. If ω_h is the highest (radian) frequency contained in t), if T is the duration of this signal, and if

$$\omega_h T\ \delta < 1, \tag{7}$$

en the Doppler effect is wholly negligible as far as correion detection of the signal is concerned. In words, (7) tes that the Doppler effect is insignificant if the product duration and highest frequency present in a signal is s than the reciprocal of δ. (In the case of radio comunication to objects moving at the velocity of earth tellites, the reciprocal of δ is of the order of magnitude ,000.)

If $\omega_h T\delta$ is equal to or greater than one, the Doppler ect produces significant changes in the received signal. the transmitted signal bandwidth ω_m is small in comrison with its center frequency ω_a, and if, further,

$$\omega_m T\ \delta < 1, \tag{8}$$

en the received signal resembles that which was transitted, except that all frequency components have been splaced by an amount $\delta\omega_a$, the so-called "Doppler ift." The discussion of wide-band signals for which (7) es not hold is beyond the scope of the present paper, is the discussion of narrow-band signals for which (8) ils.

The rest of this section introduces the analytical inciples used in the remainder of the paper and describes e Doppler effect in detail in terms of the changes proced in the Fourier spectrum of a signal.

he Doppler Effect

What is known as the Doppler effect is the consequence a (uniform) rate of change in length of the path traversed a signal in transmission. It appears as an apparent time ale stretch between transmitter and receiver.

$$t_r = \alpha t. \tag{9}$$

the transmitted signal is

$$s_t(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_t(\omega)e^{i\omega t}\, d\omega, \tag{10}$$

then the received signal is (except for a constant delay)

$$s_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_t(\omega)e^{i\omega t_r}\, d\omega, \tag{11}$$

or, except for a constant delay,

$$s_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_t(\omega)e^{i\omega\alpha t}\, d\omega = s_t(\alpha t). \tag{12}$$

If we analyze the received signal at the receiver so that we can write

$$s_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_r(\omega)e^{i\omega t}\, d\omega \tag{13}$$

we find that

$$S_r(\omega) = (1 + \delta)S_t[(\omega)(1 + \delta)]. \tag{14}$$

Normally, the transmitted signal $s(t)$ is the result of modulating a sinusoidal carrier with a signal $f(t)$. If single sideband modulation[7] is used, the transmission is best represented in terms of real rather than complex Fourier integrals. If

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\, d\omega, \tag{15a}$$

then also

$$f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} A(\omega)\cos\omega t\, df + \frac{1}{\pi}\int_{-\infty}^{\infty} B(\omega)\sin\omega t\, d\omega \tag{15b}$$

$$\left.\begin{aligned} &\text{where } A(\omega) = \text{Real part of } F(\omega)\\ &\text{and } B(\omega) = \text{Imaginary part of } F(\omega).\end{aligned}\right. \tag{15c}$$

The signal transmitted is then (by definition of "single-sideband")

$$f_t(t) \triangleq \frac{1}{\pi}\int_0^{\infty}\left[A(\omega)\cos(\omega + \omega_a)t\, d\omega + \frac{1}{\pi}\int_0^{\infty} B(\omega)\sin(\omega + \omega_a)t\right]d\omega. \tag{16}$$

By (14), the received signal is, except for a multiplicative constant,

$$f_r(t) = \frac{1}{\pi}\int_0^{\infty} A(\omega)\cos\{[\omega + \omega_a][(1 + \delta)t - \zeta]\}\, df + \frac{1}{\pi}\int_0^{\infty} B(\omega)\sin\{[\omega + \omega_a][(1 + \delta)t - \zeta]\}\, df \tag{17}$$

where a fixed delay ζ has now been added into the expression.

If the received signal is coherently demodulated, there results a time function, $g(t)$, where

[7] This is the natural method of modulating (*i.e.*, heterodyning) if the signal $f(t)$ is generated at band-pass (see Section III).

$$g^{(t)} = \frac{1}{\pi}\int_0^\infty A(\omega)\cos\{\omega(1+\delta)t + \delta\omega_a t - \omega_a\zeta - \omega\zeta\}\,d\omega + \frac{1}{\pi}\int_0^{\infty} B(\omega)\sin\{\omega(1+\delta)t + \delta\omega_a t - \omega_a\zeta - \omega\zeta\}\,d\omega. \tag{18}$$

This representation of the demodulated received signal is exact. $\delta\omega_a$ is the "Doppler shift" which will be henceforth designated by Δ. Since ζ is fixed but otherwise arbitrary, the term $\omega_a\zeta$ will be replaced by a fixed phase shift, λ. If the interval, T during which an operation is performed on $g(t)$ is such that

$$\omega\,\delta T \ll 1 \tag{19}$$

for all ω for which $A(\omega)$ and $B(\omega)$ differ appreciably from zero, then the factor $(1+\delta)$ may be replaced by unity in (18) with negligible error. [This is the basis of (8).] Assumption (19) will be made, so that (18) becomes

$$g(t) = \frac{1}{\pi}\int_0^\infty A(\omega)\cos[\omega(t-\zeta) + \Delta t - \lambda]\,d\omega + \frac{1}{\pi}\int_0^\infty B(\omega)\sin[\omega(t-\zeta) + \Delta t - \lambda]\,d\omega. \tag{20}$$

By further trigonometric manipulations, this expression takes the form

$$g(t) = \cos(\Delta t - \lambda) \times \int_0^\infty \frac{1}{\pi}[A(\omega)\cos(t-\zeta) + B(\omega)\sin(t-\zeta)]\,d\omega - \sin(\Delta t - \lambda) \times \int_0^\infty \frac{1}{\pi}[A(\omega)\sin(t-\zeta) - B(\omega)\cos(t-\zeta)]\,d\omega. \tag{21}$$

Now the upper integral in this last expression is simply the inverse transform of the original signal delayed by an amount ζ.

$$\frac{1}{\pi}\int_0^\infty [A(\omega)\cos(t-\zeta) + B(\omega)\sin(t-\zeta)]\,d\omega = f(t-\zeta). \tag{22}$$

In the second integral in (21), every frequency component in the first integral has been shifted in phase by 90° without altering its magnitude. That is to say, every frequency component of $f(t-\zeta)$ has been shifted in phase by 90°, turning $f(t-\zeta)$ into its "conjugate function" $\hat{f}(t-\zeta)$. The properties and uses of conjugate functions have been discussed by Gabor [10], Dugundji [11], and others. A conjugate function $\hat{h}(t)$ is, in general, related to an $h(t)$ by the Hilbert Transform

$$\hat{h}(t) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{h(x)}{x-t}\,dx \tag{23}$$

(Cauchy principle value near $x = t$). Thus, in performing the inverse Fourier transform (22), the second integra becomes $\hat{f}(t-\zeta)$ and we obtain for the frequency-shifte signal after demodulation

$$g(t) = \cos(\Delta t + \lambda)f(t-\zeta) - \sin(\Delta t - \lambda)\hat{f}(t-\zeta). \tag{24}$$

Detection and Filtering of Doppler Signals

In (24) we have an expression for a signal which ha been subjected to a delay and a frequency shift. In genera the delay ζ and the phase shift λ as well as the Dopper must be regarded as initially unknown parameters of th transmission path. The advantage of the matched filte method of computing the signal estimation integrals (1) i that it does so continuously in time, so that the time dela ζ only introduces a corresponding delay in the output c the signal decision circuits. We shall take advantage c this system invariance with respect to delay to simplif the mathematical expressions by leaving out fixed delay (including the delay ζ)[8] in the knowledge that they ma be subsequently reinserted if necessary. Thus, we sha use as the expression for the Doppler shifted signal

$$g(t) = \cos(\Delta t - \lambda)f(t) - \sin(\Delta t - \lambda)\hat{f}(t). \tag{25}$$

On the other hand, we cannot drop with impunity th arbitrary phase shift λ. Even when the transmission take place over a fixed path with Δ equal to zero, the absolut radio frequency phase shift along the transmission path i difficult to determine. When the phase shift λ is not know signal decision theory[9] [1], [2] requires that we comput for each different (except for phase shift λ) possibl noise-free signal signal $s_k(t)$

$$J_k = \sqrt{\left(\int y(t)s_k(t)\,dt\right)^2 + \left(\int y(t)\hat{s}_k(t)\right)^2}. \tag{26}$$

Here $y(t)$ is, as before, the actual received signal as dis turbed by additive white Gaussian noise. The decision a to which signal, if any, is present is then based on comparison of the integrals J_k with each other and fixe thresholds.

A matched filter system is one in which the integra required in (26) or (1) are computed by passing th corrupted signal $y(t)$ through linear filters whose transie responses are (except for a delay) $\{s_k(-t)\}$ and $\{\hat{s}_k(-t)\}$. The output of such a filter is the cross-correlation functio $\varphi_{sy}(t)$[10]

$$\varphi_{sy}(t) = \int_{-\infty}^\infty y(x)s(x-t)\,dx. \tag{27}$$

$\varphi_{sy}(0)$ is then the integral required in making the signa

[8] In the case of the Doppler effect, ζ is of course not a *fixed* dela The various assumptions [such as (19)] under which our prese analysis of the Doppler effect is valid are tantamount to the assum tion that ζ may be regarded as a fixed delay in the process of sign detection.

[9] Results given the remainder of this section will be state rather than demonstrated. An excellent survey of the releva theory is the review paper by G. Turin [13] in this issue.

[10] The more common definition of the correlation function h $+t$ rather than $-t$ in the integral. Clearly, this change of sig makes no essential difference in the basic properties of the functio

lecisions. If $s(t)$ is set equal to $f(t)$ and, if further, $y(t)$ is Doppler translate of $f(t)$, as given by (25), uncorrupted y noise, then the filter output is called an "Uncertainty Function" or "Ambiguity Function" [1], [3], [4], [5], 7], [16], $X_1(t, \Delta)$:

$$X_1(t, \Delta) = \int_{-\infty}^{\infty} f(x - t)[f(x) \cos (\Delta x - \lambda) - \hat{f}(x) \sin (\Delta x - \lambda)]\, dx. \tag{28}$$

Similarly, we have for the second integral in the expression or J

$$X_2(t, \Delta) = \int_{-\infty}^{\infty} \hat{f}(x - t)[f(x) \cos (\Delta x - \lambda) - \hat{f}(x) \sin (\Delta x - \lambda)]\, dx. \tag{29}$$

A very much more compact formulation of these filter output functions is obtained by using the complex output $X(t, \Delta)$ of which $X_1(t, \Delta)$ is the real part and $X_2(t, \Delta)$ the imaginary part:

$$\begin{aligned} X(t, \Delta) &= X_1(t, \Delta) + jX_2(t, \Delta) \\ &= \tfrac{1}{2} \int \overline{\mathfrak{F}(x - t)}\, \mathfrak{F}(x) \epsilon^{i(\Delta x - \lambda)}\, dx \end{aligned} \tag{30}$$

where

$$\mathfrak{F}(x) = f(x) + j\hat{f}(x).$$

The signal weight J computed by the matched filters is then given as a function of time by the magnitude of $X(t, \Delta)$. It will be noted that this magnitude is independent of the initial phase λ:

$$\begin{aligned} | X(t, \Delta) | &= \left| e^{-j\lambda} \int_{-\infty}^{\infty} \overline{\mathfrak{F}(x - t)}(x) e^{i\Delta x}\, dx \right| \\ &= \left| \int_{-\infty}^{\infty} \overline{\mathfrak{F}(x - t)}\, \mathfrak{F}(x) e^{i\Delta x}\, dx \right|. \end{aligned} \tag{31}$$

Since generally Δx goes through many cycles during the duration of $f(x)$ or $\mathfrak{F}(x)$, we can simplify the expressions for the filter outputs little further except in the case $\Delta = 0$. In that case we have

$$\begin{aligned} X_1(t, 0) &= \cos \lambda \int f(x - t) f(x)\, dx + \sin \lambda \int f(x - t)\hat{f}(x)\, dx \\ X_2(t, 0) &= \cos \lambda \int \hat{f}(x - t) f(x)\, dx + \sin \lambda \int \hat{f}(x - t)\hat{f}(x)\, dx. \end{aligned} \tag{32}$$

It is known [11] that all the integrals in (32) may be expressed in terms of the correlation function $\varphi_{ff}(t)$ and its conjugate $\hat{\varphi}_{ff}(t)$. In particular,

$$\begin{aligned} X_1(t, 0) &= \cos \lambda \varphi_{ff}(t) + \sin \lambda \hat{\varphi}_{ff}(t) \\ X_2(t, 0) &= \cos \lambda \hat{\varphi}_{ff}(t) - \sin \lambda \varphi_{ff}(t). \end{aligned} \tag{33}$$

Finally, if we square and add these two functions to obtain J^2, we have

$$J = | \varphi_{ff}(t) |. \tag{34}$$

Thus, when there is no Doppler shift, but an initial unknown phase, the computation of J produces the same magnitude of output as would be delivered by the X_1 filter if the phase were known and equal to zero.

III. Equipment Design Principles

In the preceding sections we have assembled a theoretical background on which to base the design of practical devices for the detection of Doppler shifted signals. In carrying out practical designs it is necessary to keep in mind engineering questions which for reasons of simplicity and tractability do not (or cannot) appear in mathematical models of the detection problem. These questions can be divided into two broad categories, those which concern the choice of the signal function $f(t)$ at the transmitter and the ones which concern the means by which the presence of this signal is indicated at the receiver.

It is a general result of detection theory that the reliability with which a given signal $f(t)$ can be detected (in the presence of additive white Gaussian noise) depends only on the total energy in $f(t)$. Accordingly, the details of the waveform $f(t)$ are dictated by considerations [such as duration, bandwidth, or type of corresponding $X(t, \Delta)$] which are not directly related to the signal detection problem being discussed here. In fact, in the present case, technology played an important role in the selection of an $f(t)$. In the first place, it was not known at the beginning of the project exactly what signal waveform would turn out to be the most interesting. Consequently, it was decided to construct signals and filter characteristics in the time domain in such a way that the function $f(t)$ and the corresponding receiver characteristics could be varied at will. This was done with the aid of a delay line having a multiplicity of output taps at regular intervals in time. Secondly, it was desirable to generate and detect signals $f(t)$ which have a reasonably large duration-bandwidth product. A TW product of 100 was arbitrarily selected[11] as being within the then existing limits of tapped delay line technology. Finally, it was decided to generate and detect the signals at "band-pass" rather than at "video" to take advantage of a substantial saving in the number of filtering operations required. As we shall see later, this saving is a consequence of the simplicity of obtaining the conjugate function $\hat{f}(t)$ at bandpass.

With respect to signal detection, it is not the object of this paper to attempt a rigorous presentation of the principles by which equipment might be assembled to carry out the computation and comparison of the signal

[11] If an $f(t)$ which has an $X(t, \Delta)$ that is uniformly low away from the origin ($t = 0$), ($\Delta = 0$) is desired, subsequent theoretical developments [6], [7] indicate that $TW = 127$ would have been a more logical choice for the magnitude of the TW product of the signal.

estimation integrals (1) and (26). Indeed, many of the features of any such equipment are necessarily matters of engineering taste. Instead, we shall point to several important general problems which arise in the construction of equipment to implement the detection process and indicate the manner in which they were met in a particular case. Three such questions are:

1) How should one deal with a continuum of possible signals? If the collection of waveforms that may be expected by a receiver is finite, then an optimum guess as to what signal (if any) is received (in the presence of additive white Gaussian noise) involves a comparison of the actual received signal with each of these possibilities. If it is known in advance only that the frequency shift of a signal will be within certain limits, then the "ideal" receiver must perform in principle an infinitude of filtering operations, corresponding to the continuum of possible frequency shifts. This infinitude of operations cannot in general be built into practical equipment. Accordingly, one builds not an "optimum" detector but an "adequate" one, in the sense that its performance approximates within prescribed limits that of the mathematically optimum procedure.

2) How can the signal processing equipment be kept as simple as possible? When dealing with signals which are complicated, in the sense that many parameters (*e.g.*, samples) are necessary to specify them, it is difficult to avoid the fact that signal storage is generally available or accessible only in units[12] which basically store or process a single parameter. Consequently, the signal processing filters are likely to be complicated devices. If the Doppler shift can be large when compared to the reciprocal of the signal duration, many such filters will be necessary in the receiver. Especially if the transmitted signal $s(t)$ and the corresponding response of the detector filters are to be changed, it is necessary to give some engineering thought to minimizing the complexity of the resulting apparatus, even at the price of a small sacrifice in system performance.

3) What care must be exercised (and what advantage can be taken) with respect to the sensitivity (or insensitivity) of the performance of a detection system to variations in the performance or parameter values of its component parts? For example, the performance of a matched filter detection system may be very sensitive to differential drift in the phase angles associated with frequency response of individual elements in the matched filter; it may be very insensitive to the introduction of violent nonlinear distortion into the received signal-plus-noise [8].

Clearly, many other design questions than these three could be cited. It is also evident that there are many configurations of detection equipment to which they might be applied. We will be content here, however, to use these three sets of questions (in Section IV) to discuss a particular experimental apparatus that was built to demonstrate the practicability of successfully detecting signals subject to a range of frequency shift that is large compared with the reciprocal of their duration.

IV. Matched Filter Detection System

The basic philosophy of the system described here is to build a set of filters each "matched" to a significantly different frequency translate $s(t)$ of a band-pass signal $f(t)$. The signal $s(t)$ corrupted by additive white Gaussian noise is then applied to these filters. The largest of the envelopes of the resulting outputs is compared with a threshold, and the signal is announced as present when the threshold is exceeded (see Fig. 1). This detection system is known[13] to be very close in its performance to the mathematically optimum detection procedure, if the particular signals for which instrumentation is provided are the only ones possible. Since these are not the only frequency translates that are possible, one must determine the number of filters that will be required in the matched filter bank if the system of Fig. 1 is to perform adequately. This is basically a question of determining what happens to the output of a given matched filter when it is excited by a signal whose frequency translation differs by Δ cycles per second from the one for which it was designed. It is a difficult question to answer without some further information about $s(t)$. Let us assume that $s(t)$ is a narrow band signal whose bandwidth is approximately W cps and whose duration is approximately T seconds. Let the product TW be large, as would be the case with a T-second sample of shot noise of bandwidth W. Then the output of a filter correctly "matched" to $s(t)$ will be the autocorrelation function of $s(t)$. Because $s(t)$ is a narrow-band signal, this output will also be narrow-band and will resemble a sinusoid at the band-center frequency which is phase and amplitude modulated at a rate corresponding to the bandwidth W [11].

The envelope of such an output is sketched in Fig. 2. This autocorrelation function has a central "spike" whose amplitude is proportional to the signal energy and whose duration is roughly equal to the reciprocal of the signal bandwidth. For T seconds on either side of this central spike there extends a region of lower output which has been called "side lobe level" or, more simply, "hash." The rms value of this "hash" in the matched filter output generally lies about $1/\sqrt{TW}$ below the peak of the central spike.[14] If noise as well as signal is present at the filter input, then the SNR in the output of the matched filter (at the time of the peak of the central spike) is E/N_0, where E is the energy of the signal and N_0 is the added

[12] Examples of such units are the flip-flop, the amplitude and phase of a frequency component as obtained by a resonant circuit, or the magnitude of a signal element as represented by the weighting assigned to a tap on a delay line.

[13] A full discussion of this point is beyond the scope of the present paper. See [1], [2], [3] and the paper by G. Turin in this issue for a fuller exposition.

[14] Exceptions to such general statements about the hash level are easy to find, See [4], [5], [6], [13] for fuller discussion.

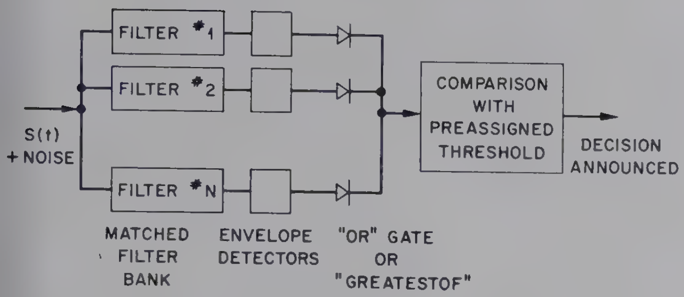

Fig. 1—Matched filter section system.

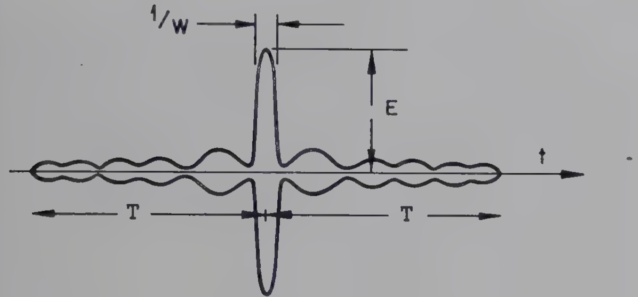

Fig. 2—Envelope of matched filter output.

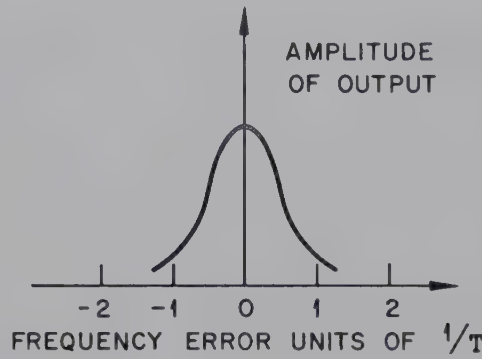

Fig. 3—Effect of frequency mismatch in filter.

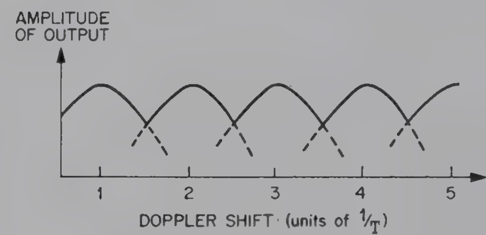

Fig. 4—Filter bank performance.

white Gaussian noise power per (single-sided) cycle of bandwidth at the filter input.

Suppose now that the matched filter is excited by a different frequency translate of $f(t)$ than the one for which it was designed. In general, the magnitude of the central spike will decrease and the level of both the noise and the hash[14] in the filter output will remain about the same. The SNR in the filter output thus falls off as the square of the decrease in the amplitude of the spike. The precise manner in which the spike output of the matched filter varies with frequency mismatch depends on the form of the envelope of $s(t)$. In general, if $s(t)$ has an effective duration of T seconds, the central spike falls to a low value in a frequency mismatch interval comparable with $1/T$, as shown in Fig. 3. We will accordingly assume that the maximum practicable separation of the frequency translates to which members of the filter bank are matched is $1/T$ cps. Enough filters must be provided to cover the range of Doppler shifts which the signal may suffer. If the filter bank is constructed using the maximum interval, the signal available to operate the decision circuits varies with Doppler shift in the manner shown by the solid line in Fig. 4. The dotted lines represent the continuation of the individual channel outputs as the largest output shifts from channel to channel with changes in Doppler shift. With this maximum separation of filter translation frequencies, there will be a degradation of from 3 to 5 db [depending on the details of the envelope of $s(t)$] in the performance of the system when the Doppler shift lies halfway between those shifts to which adjoining filters are matched. That is to say, 3 to 5 db more signal will be required to produce a given level of detector performance when the Doppler lies halfway in between than when it equals that to which a given filter is matched. If the number of filters is doubled, so that the filters are matched to frequency translates of $s(t)$ in intervals of $1/2T$ cycles, the variation of system performance with Doppler shift can in principle be reduced to about 1 db or less.

Although we shall not prove the point here, it is worthwhile to remark that if $s(t)$ is strictly limited to the duration T, then one need actually construct little more than a minimal bank of filters matched to frequency translates of intervals of $1/T$, in the sense that other filter outputs that one may desire can be realized as linear combinations of the outputs of the minimal bank.

The second problem to be dealt with in constructing a matched filter system of the scope under discussion here is that of keeping the complexity of the required filtering arrangements within reasonable bounds.

In Section II, *Detection and Filtering of Doppler Signals,* it was pointed out that for each different signal of unknown phase it is necessary to process the actual received waveform in two filters, one matched to the signal and the other matched to the conjugate function of that signal. In the present case that signal, $s_k(t)$, is a frequency translate of an initial signal $f(t)$ by a (radian) frequency Δ:

$$s_k(t) = \cos(\Delta t) f(t) - \sin(\Delta t)\hat{f}(t) \tag{35}$$

and the corresponding conjugate function is [11]

$$\hat{s}_k(t) = \cos(\Delta t)\hat{f}(t) + \sin(\Delta t) f(t). \tag{36}$$

At the beginning of this section it was pointed out that one set of these filters must be provided for each $1/T$ frequency interval ($2\pi/T$ in radian frequency) of range of possible Doppler shift; indeed, double this number might be necessary. In the present case it was desired to cover a range of Doppler frequencies some 17 units of $1/T$ wide. Thus one is faced with the prospect of constructing some 34 different filters each having a different, complicated transient response with TW product of 100. To cover

negative as well as positive Doppler shifts the number of filters must be doubled; if the filters are to be placed at half intervals of $1/T$ as suggested, this number must be doubled once again. Thus, unless we are careful we find ourselves constructing some 136 filters, each of which contains the equivalent of 100 inductors and capacitors, without yet having considered the requirement of being able to change the system to accommodate an arbitrary $f(t)$. Clearly, minor compromises with the "ideal" detection procedure are acceptable if they result in substantial savings in the number of filter components required. In point of fact, the entire filter task was handled in one 100-tap tapped delay line and a resistor matrix made up of some 4000 resistors. We shall now describe this delay line filter and the nature of the compromises which permit it to be used efficiently.

The first reduction that can be made in the complexity of the filtering system is to eliminate the need for the filter matched to $\hat{s}_k(t)$ by generating and processing signals at band-pass. In general, the Hilbert transform of a signal bears no direct simple relationship to the signal itself [other than the defining integral (23) or the 90° phase shift relation]. For example, the conjugate function to a sharp rectangular pulse is a pulse of an entirely different waveshape which falls to zero asymptotically only as $1/t$. On the other hand, it is well-known that a band-pass signal can be represented as a sine wave at the band-center frequency ω_0 whose envelope R and phase θ are slowly varying functions of time:[15]

$$f(t) = R(t) \cos [\omega_0 t - \theta(t)]. \tag{37}$$

Inasmuch as the Hilbert transform of a sinusoid is another sinusoid shifted in phase by 90°, it is reasonable to expect that $\hat{f}(t)$ can be written

$$\hat{f}(t) = -R(t) \sin [\omega_0 t - \theta(t)] + Q(t) \tag{38}$$

where $Q(t)$ is an error term whose magnitude is small compared with that of $R(t)$. In the Appendix, it is shown that this is indeed the case, and that the relative size of $Q(t)$ is at most of the order of magnitude of the percent bandwidth of the signal.

If we consider now the output of the filter matched to $s_k(t)$ to be $\psi_k(t)$, Dugundji [11] has shown that with a common input the output of a filter matched to $\hat{s}_k(t)$ is $\hat{\psi}_k(t)$. Hence, we have for the signal comparison integral function J_k

$$\begin{aligned} J_k &= \sqrt{[\psi_k(t)]^2 + [\hat{\psi}_k(t)]^2} \\ &\doteq \sqrt{\{R_\psi(t) \cos [\omega_0 t - \theta(t)]\}^2} \\ &\quad + \sqrt{\{R_\psi(t) \sin [\omega_0 t - \theta(t)]\}^2} \\ &= | R_\psi(t) |. \end{aligned} \tag{39}$$

[15] See Dugundji [11] and also S. O. Rice, "Mathematical analysis of random noise," in "Noise and Stochastic Processes," Dover Publications, New York, N. Y., pp. 133–294. Generally $R(t)$ is considered to be defined to be always positive. We note that here $R(t)$ must be permitted to change sign if the phase function $\theta(t)$ is to be truly "slowly varying." Otherwise $\theta(t)$ will exhibit discontinuities of π radians which would vitiate the analysis given in the Appendix.

where R_ψ is the envelope of the output of the matched filter. Although the envelope of a narrow-band filter should in principle be found by a process indicated in (39), it is much easier to do envelope detection with a full wave rectifier and a low-pass filter. This was in fact done, thus eliminating the need for a filter matched to $\hat{s}_k(t)$.

It is now clear that at band-pass the $\hat{f}(t)$ which appears in the second term of the expression (35) for $s_k(t)$ is also related to $f(t)$ by a simple 90° phase shift. Before we try to use this fact to effect further simplifications in the detection process, it is convenient to introduce the delay line synthesis of the signal $f(t)$.

Consider the cascade of an ideal delay line and a bandwidth determining filter shown in Fig. 5. This ideal delay line has N steps of delay each equal to τ seconds. The filter has a transient response, such as that shown in Fig. 6, which is a simple band-pass signal lasting approximately τ seconds. In this cascade arrangement the output of the nth tap is $u_n(t)$ where

$$u_n(t) = u(t - n\tau). \tag{40}$$

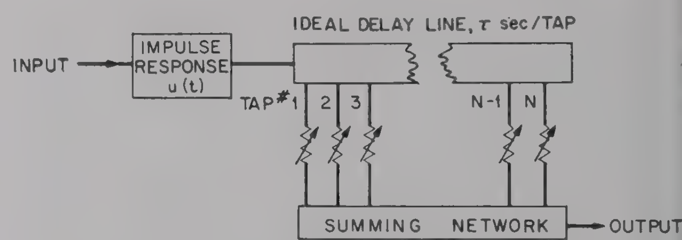

Fig. 5—Delay line model of signal generation.

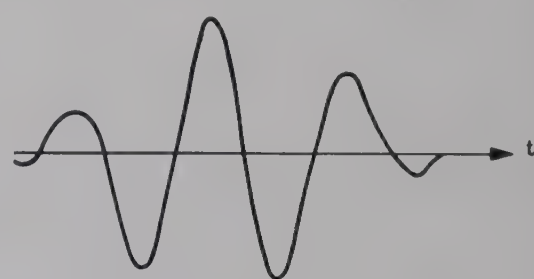

Fig. 6—Elementary signal $u(t)$.

At each tap we provide a means of adjusting the tap output (shown schematically as a variable resistor) by a weight a_n. Thus, it is possible to synthesize a general signal of the form

$$f(t) = \sum_{1}^{N} a_n u_n(t - n\tau). \tag{41}$$

Since the signal is being formed at band-pass the phase as well as amplitude at each tap output should be (and was) adjustable. Under these circumstances one can closely approximate any $f(t)$ which lasts less than $N\tau$ seconds and which has a bandwidth of about $1/\tau$ cps.[16]

[16] This last statement is basically a statement of the Sampling Theorem. The kind of generalization we use here has been discussed by Gabor [10] and Lerner, *op. cit.*, footnote 2.

The matched filter to the $f(t)$ given above is one whose transient response is (except for a fixed delay)

$$f(-t) = \sum a_n u(-t - n\tau). \tag{42}$$

This filter could in principle be constructed just as was that which generates $f(t)$, *i.e.*, from a pulse shaping network in cascade with a tapped delay line. In this case, the transient response of the pulse shaping network should be (except for a fixed delay) equal to $u(-t)$ instead of $u(t)$, and the taps of the delay line should be weighted with the same amplitude, the negative of the phase angle, but in the reverse order as in forming $f(t)$. In fact, the weights at the taps of the experimental filter were set up in the manner just described, but the transient response of the pulse shaping network was $u(t)$ instead of $u(-t)$. To this end, $u(t)$ was designed to be approximately symmetric in time with respect to its mean time of occurrence [*i.e.*, $u(t) \approx u(-t)$ except for a constant delay] so that the mismatch loss from this cause would be negligible.

We wish to use the tapped delay line not only to produce $f(t)$ but also to set up the transient response characteristics of the various matched filters. If we substitute the expression (41) for $f(t)$ as a sum of $u_n(t)$ in the expression (35) for Doppler shifted signal we obtain

$$\begin{aligned} s(t) &= f(t) \cos \Delta t - \hat{f}(t) \sin \Delta t \\ &= \sum a_n u_n(t) \cos \Delta t - \sum a_n \hat{u}_n(t) \sin \Delta t. \end{aligned} \tag{43}$$

Now, in principle, it may be possible to express this new function in terms of a set of $u_n(t)$ only[17]

$$s(t) = \sum b_n u_n(t) + \sum \beta_n \hat{u}_n(t). \tag{44}$$

However, for the sake of simplicity in changing the signal function $f(t)$, this course was not followed. Instead, it was assumed that the Doppler shift was sufficiently small so that the functions $\cos \Delta t$ and $\sin \Delta t$ varied but little during a time interval τ equal to the duration of $u(t)$. Under these circumstances, (43) may be rewritten

$$\begin{aligned} s(t) \doteq \sum a_n u_n(t) \cos n\, \Delta\tau \\ - \sum a_n \hat{u}_n(t) \sin n\, \Delta\tau. \end{aligned} \tag{45}$$

The only other factor needed to pass to the form in which the matched filter was realized is the observation that $\hat{u}(t - n\tau)$ differs from $u(t - n\tau)$ principally in that the carrier under the envelope is shifted by 90° in phase. Accordingly, the second sum in (45) can be first obtained as

$$\sum = \sum a_n u_n(t) \sin n\, \Delta\tau \tag{46}$$

and later be shifted in phase by 90° for addition to the first sum at the filter output.

It should now be observed that each term in the sums (45) has two weights, one due to the signal $s(t)$, *viz.* a_n, and one due to the Doppler shift, *viz.* $\cos n\Delta\tau$ or $\sin n\Delta\tau$. Accordingly, the matched filter may be realized in the form of Fig. 7. Each of the tap outputs is first weighted in proportion to the a_n, in reverse order to that used in forming $s(t)$. (Resistors are shown as the weighting elements in the figure. In fact, both positive and negative outputs are available from the delay line taps so that real resistors can be used in the weighting.) Then these outputs are further weighted according to the cosine and sine functions corresponding to a given Doppler shift. For zero Doppler shift, all the additional weighting resistors to the "zero Doppler bus" are of course equal. The first Doppler shift was taken equal to $1/T$, the reciprocal of the duration of the signal $s(t)$. Accordingly, the weighting nth tap bus to the first cosine Doppler bus becomes (in accordance with the order reversal of taps in the matched filter) $\cos \Delta\tau(N - n)/T$; the weights to the first sine bus follow a similar pattern. Filters matched to further frequency translates of $s(t)$ are added to the structure by adding the weighting elements to a corresponding set of sine and cosine busses.

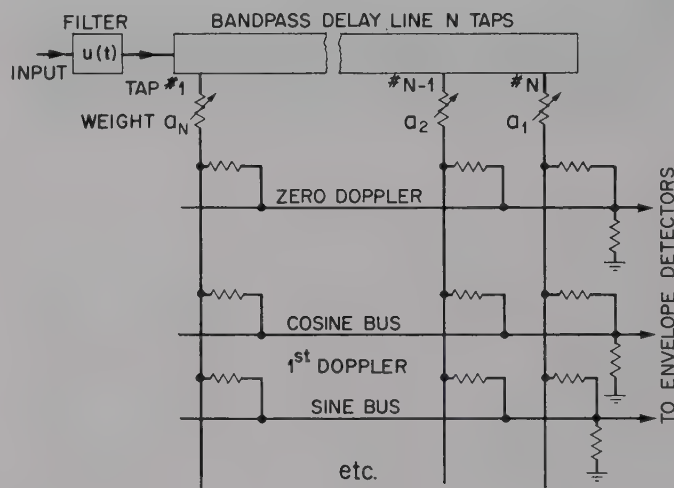

Fig. 7—Matrix filter.

Thus, an entire bank of 18 Doppler filters, including zero Doppler, was constructed from a single 100 tap delay line. The resulting weighting matrix contained nearly 4000 resistors. Nevertheless, the function $f(t)$ to which this system responded could be easily changed by varying the 100 initial tap weights. Part of the resistor matrix is shown in Fig. 8.

In the previous sections we have mentioned several engineering departures that were made from the mathematically "optimum" signal processing, on the grounds that the resulting degradation in system performance was negligible. Two further liberties were taken with the mathematical model. First, in discussing (46) it was pointed out that the output of the sine bus should be shifted in phase by 90° and added to that of the cosine bus for a given Doppler channel to produce the matched filter output. By methods beyond the scope of this paper it can be shown that very little loss in signal detection performance of the decision circuitry results if the (band-

[17] See Lerner, *op. cit.*, footnote 2.

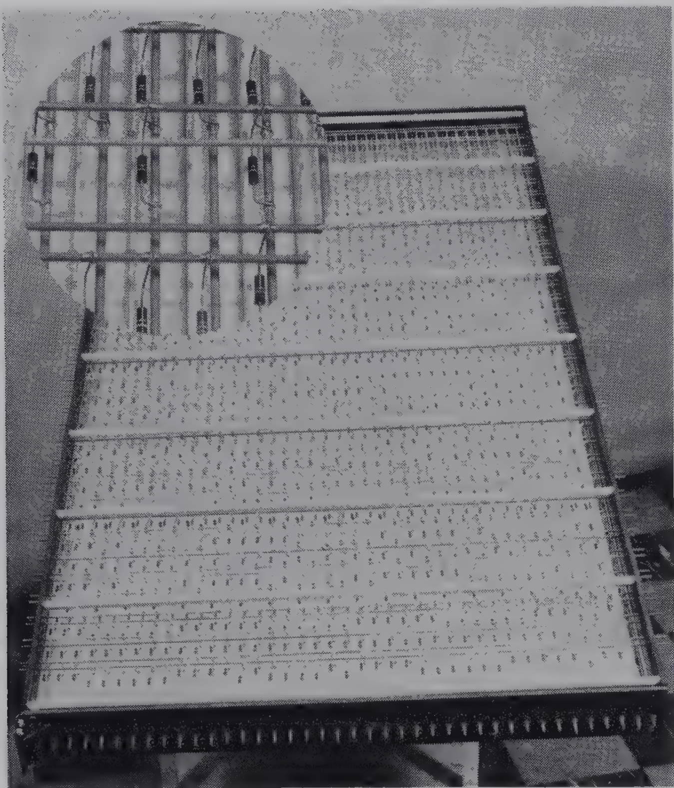

Fig. 8—Resistor weighting matrix.

pass) outputs of the sine and cosine channels are simply full wave rectified and then added together. This sum of rectified signals closely approximates the envelope of the signal that would have been obtained if the mathematically correct phase shifting and summation had been used.[18] It further eliminates the sensitivity of the detection process to the sign of the Doppler shift; thus one set of filters suffices for both positive and negative Doppler.

Whereas the success of the various approximations and compromises mentioned up to this point has been independent of the TW product of $s(t)$, the system properties and sensitivities to be discussed in the remainder of this section are definitely those of signals with a large duration-bandwidth product.

The second compromise referred to above falls in this category. Namely, to stabilize the performance of the decision circuitry with respect to wild swings in input signal level, the entire matched filter system was preceded by a band-pass amplifier having sufficient gain to be driven nearly to saturation by its own internal noise. Thus, for most signals this amplifier was effectively an infinite peak clipper, preserving only the zero-axis crossings of an input signal. With TW of the order of 100, this violent nonlinearity makes little observable difference in either the general waveform of the matched filter outputs or in the performance of the system as a detector of signals.[19]

[18] This particular feature is due to a suggestion made by D. J. Gray.

[19] See, for instance, Manasee *et al*, [8].

It is beyond the scope of the present paper to discuss in detail the nature of the tolerances that can be permitted in the component parts of a delay line filter system for detecting signals with high duration-bandwidth product.[20] However, the following comments are in order.

1) Relatively gross random errors can be tolerated in the amplitude or phase angle associated with the weighting factor at a given delay line tap or at a given cross point in the signal processing matrix. If there are 100 taps making roughly equal contribution to the signal, the complete elimination of the output of any one tap will scarcely make a noticeable difference (1 per cent) in the amplitude of a matched filter output. More specifically, if the tolerance on the matrix components or phase angles is ϵ in a signal detection system that requires an N tap delay line, then the resulting variability introduced in the matched filter outputs is of the order of $\epsilon/\sqrt{N}$.

2) The tolerance of the system to internal reflections and spurious response in the delay line is low. It becomes more critical as the TW product (number of delay line taps) of the signal is increased. The spurious responses to an impulse in the delay line should be below the main pulse by 10 log TW plus 10 to 20 db.

3) The sensitivity of the output of *a single filter* to systematic drift in the total delay of the delay line increases with the product of signal duration and band-center frequency. Such drifts (due to temperature changes, for example) are equivalent to time scale changes; they are not too important if a *bank* of filters is employed for *detection* purposes only. They are critically important if the output of a particular filter is relied on to give an accurate *measurement* of the Doppler shift.

V. Experimental Results

An eighteen channel system was constructed, designed in accordance with the principles outlined above. The delay line filters made use of a tapped magnetostrictive acoustic wire delay line developed at Lincoln Laboratory by S. Autler, G. Pettengill and the author. A schematic diagram of the operation of this delay line is shown in Fig. 9.

An electric signal was transformed into acoustic energy in a ceramic piezo-electric input transducer. An acoustic matching transformer connected this transducer to a thin rod of magnetic material around which center tapped output coils were placed at regular intervals. A magnetic bias field was supplied at each coil. As the sound field passes through the rod it alters its local magnetic properties. The corresponding changes in the flux due to the bias field are converted into electric signals at the output coils. The signal at a given coil is thus delayed with respect to the original electrical input by the amount of time required for the acoustic energy to propagate down

[20] See R. M. Lerner, B. Reiffen, and H. Sherman, "Delay-line specifications for matched-filter communications systems," IRE Trans. on Component Parts, vol. CP-6, pp. 263–268; December 1959. Also R. M. Lerner, "Delay lines for use in signal detection," IRE Trans. on Ultrasonics Engineering, to be published.

the thin rod to that point (at the rate of about 5 km/second). 100 taps were used on the line.

The weighting matrix was made up of 5 per cent tolerance resistors, whose nominal values were rounded off from calculations in such a way that only a dozen or so different resistance values were used in approximately equal numbers.

The magnitudes of the envelopes of all the filter outputs were compared in a cathode follower "greatestof" circuit shown in Fig. 10. (These cathode followers are operating basically as a diode "or" gate.) The largest value of the envelope in any channel at each instance was thereby delivered to the decision circuits. Since the Doppler filters were spaced by intervals of $1/T$ in frequency translation, the outputs of a second set of "interdoppler" filters was synthesized by linear combinations of Doppler filter outputs, as was suggested in the discussion of Fig. 4.

To test the behavior of the system, the signal $s(t)$ was synthesized in one delay line filter and detected in another. Between the filters, frequency shifts could be introduced, attenuation could be inserted, and noise added, as shown in Fig. 11. Ideally, as the "Doppler" shift added to $s(t)$ is varied, the amplitude of the spike delivered to the decision circuits by the "greatestof" output should vary as shown in Fig. 4. The signal appearing on the "greatestof" output for one value of Doppler, with an arbitrary $s(t)$ is shown in a tracing of a photograph in Fig. 12. A measurement of the actual variation in the height of the central spike with Doppler frequency is given in Fig. 13.

In the presence of additive noise, a detection system can make two kinds of errors:

1) A signal is missed when its presence should have been indicated.

2) The presence of a signal is announced when, in fact, none in there.

By varying thresholds in the decision circuits it is generally possible for each given level of input signal to make either kind of error small at the expense of the other. The performance of a signal decision system is best assessed by a measurement of its susceptibility to errors or both types and by the comparison of these results with theory. Such a set of measurements was made on the equipment described above, covering a range of input SNR's and decision circuit arrangements for which the probability of a type 1 error was low to moderate while that of a type 2 error was very low.

When the signal to be detected had no Doppler shift, about 3.0 db more signal had to be supplied to reach a

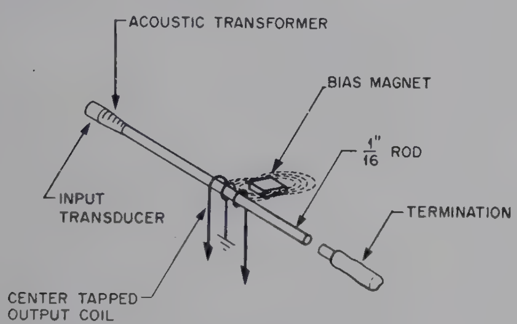

Fig. 9—Magnetostriction delay line.

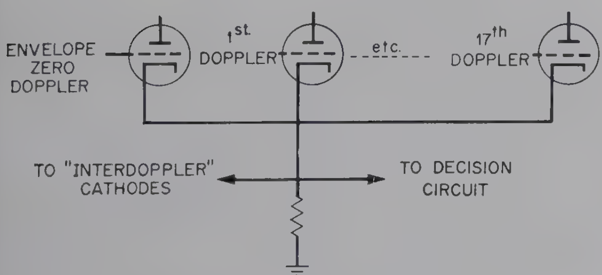

Fig. 10—"Greatestof" circuit.

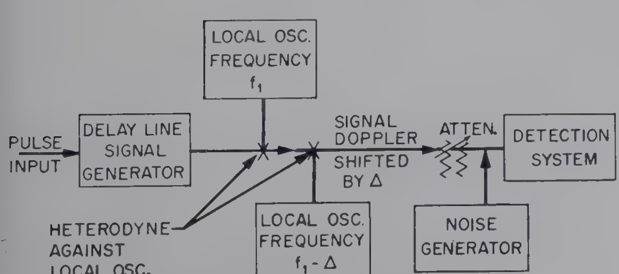

Fig. 11—Signal detection test.

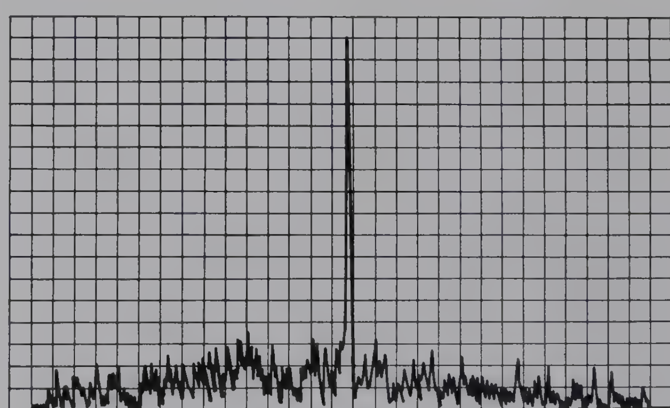

Fig. 12—Matched filter system output.

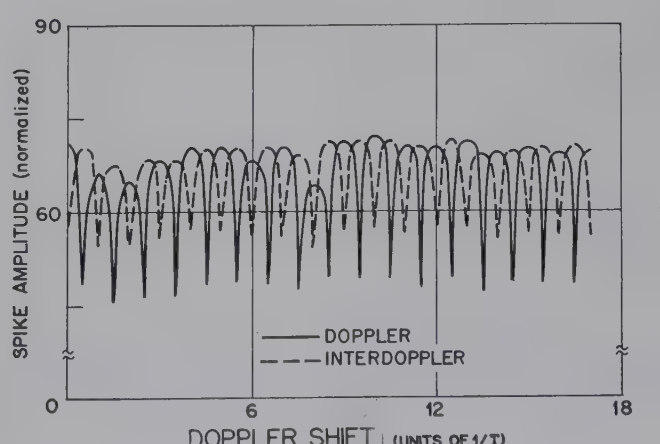

Fig. 13—System response as a function of Doppler shift.

given level of performance than would have been required by a theoretically ideal detector under the same test conditions. Of this 3 db of degradation, 0.8 db was shown to be caused by the limiter; the rest was caused by dispersion in the delay lines.[21] The degradation of the system when detecting a signal with arbitrary Doppler shift was typically 2.2 db greater, for a total of about 5.2 db. Of this additional degradation, about 1.2 db can be ascribed to nonuniformity in the Doppler channel gain functions. The remainder is caused by the approximations that were made in obtaining the actual filters from the theoretical matched filter characteristics.

VI. Conclusion

The key feature of the matched filter detection system which has just been described is the use of the tapped delay line and its associated resistor weighting matrix to synthesize in parallel a large number of filter characteristics. In the particular equipment that was constructed, it is also true that this delay line was the source of most of the difference between the performance actually observed and the performance to be expected from an "ideal" signal detector. Consequently, two questions arise with respect to the use of this kind of a matched filter system. First, in particular, could the performance of the delay lines of the system be made better by improvements in delay line technology? Secondly, in general, how does the delay line-matrix method of matched filter synthesis compare with other systems of producing and processing complicated signals? The first question can be easily answered in the affirmative. Delay line technology has reached the point where both low-pass lumped-element lines and acoustic band-pass lines suitable for producing filter characteristics with duration bandwidth products of several hundred are commercially available. These lines are free of the dispersion difficulties which plagued the experimental apparatus described above. Similarly, substantial improvements have also been made in the techniques for constructing the weighting matrices.

With respect to the second question, we remark that the usefulness of any technique of filter construction depends critically on the availability of a corresponding procedure for designing signals with prescribed characteristics. The method of filter construction used here is generally applicable where it is necessary to set up a large number of complicated filter characteristics corresponding to initially arbitrary waveforms. In any particular application another form of signal synthesis may be more appropriate.

The frequency-domain dual of the filtering system described here is a bank of contiguous narrow-band filters. In principle, at least, such a filter bank could b used in the frequency-domain synthesis of a filte characteristic in essentially the same way the delay lin is used in the time domain. In point of fact, there ar practical difficulties in the design and application of suc band-pass filter banks that make the delay line metho (initially, at least) more attractive.

The equipment described here proved to be simple, i tedious, to build, and it operated reliably. The use of tapped delay line and weighting matrix is thus establishe as a valuable method for constructing an array of compli cated matched filters.

[21] For convenience in mounting, the delay line was divided into a cascade of six shorter sections. The resulting sequence of input and output transducers produced significant amounts of distortion in the signal spectrum in the later sections of the line.

Appendix

The Hilbert Transform of a Narrow-Band Signal

We wish to derive the formulas given in Section II fo the Hilbert transform of a narrow-band signal. We writ a narrow-band function of time, $f(t)$, as the product of a envelope and a phase modulated carrier:

$$f(t) = R(t) \cos [\omega_0 t + \theta(t)] \qquad (47$$

where ω_0 is the center frequency of the band and wher $R(t)$ and $\theta(t)$ are slowly varying functions of time. $R(t)$ i allowed to change sign so that $R(t)$ and $\theta(t)$ are both continuous. Then by (26) we have

$$\begin{aligned} f_c(t) &= 1/\pi \int_{-\infty}^{\infty} \frac{d\sigma}{t-\sigma} R(\sigma) \cos [\omega_0\sigma + \theta(\sigma)] \\ &= 1/\pi \int_{-\infty}^{\infty} \frac{d\sigma}{t-\sigma} \cos \omega_0\sigma R(\sigma) \cos \theta(\sigma) \\ &\quad - 1/\pi \int_{-\infty}^{\infty} \frac{d\sigma}{t-\sigma} \sin \omega_0\sigma R(\sigma) \sin \theta(\sigma). \qquad (48 \end{aligned}$$

Now let $R(\sigma) \cos \theta(\sigma) = K(\sigma)$

$$R(\sigma) \sin \theta(\sigma) = J(\sigma) \qquad (49$$

then

$$\begin{aligned} f_c(t) = 1/\pi \int_{-\infty}^{\infty} \frac{d\sigma}{t-\sigma} \cos \omega_0\sigma K(\sigma) \\ - 1/\pi \int_{-\infty}^{\infty} \frac{d\sigma}{t-\sigma} \sin \omega_0\sigma J(\sigma). \qquad (50 \end{aligned}$$

We may now expand $K(\sigma)$ and $J(\sigma)$ in Taylor's serie around $\sigma = t$

$$\begin{aligned} f_c(t) = 1/\pi \int_{-\infty}^{\infty} \frac{d\sigma}{t-\sigma} \cos \omega_0\sigma [K(t) + (\sigma - t)K'(t) \\ + \frac{(\sigma-t)^2}{2!} K''(t) + \cdots] \\ - 1/\pi \int_{-\infty}^{\infty} \frac{d\sigma}{t-\sigma} \sin \omega_0\sigma [J(t) + (\sigma - t)J'(t) \\ + \frac{(\sigma-t)^2}{2!} J''(t) + \cdots]. \qquad (51 \end{aligned}$$

(51) may now be integrated term by term. The first terms give

$$
\begin{aligned}
f_c(t) &= 1/\pi \int_{-\infty}^{\infty} \frac{d\sigma}{t-\sigma} \cos \omega_0\sigma K(t) \\
&\quad - 1/\pi \int_{-\infty}^{\infty} \frac{d\sigma}{t-\sigma} \sin \omega_0\sigma J(t) + Q \\
&= -\sin \omega_0 t K(t) - \cos \omega_0 t J(t) + Q \\
&= -[\sin \omega_0 t \cos \theta(t) R(t) \\
&\quad + \cos \omega_0 t \sin \theta(t) R(t)] + Q \\
&= -R(t) \sin [\omega_0 t + \theta(t)] + Q. \qquad (52)
\end{aligned}
$$

This is a function with the same envelope as $f(t)$ but with carrier shifted in phase by 90°. The other terms in the series involve integrals of the form,

$$
\begin{aligned}
1/\pi \int_{-\infty}^{\infty} & (\sigma - t)^n \cos \omega_0 \sigma \, d\sigma \\
&= \frac{\cos \omega_0 t}{\pi} \int_{-\infty}^{\infty} \nu^n \cos \omega_0 \nu \, d\nu \\
&\quad - \frac{\sin \omega_0 t}{\pi} \int_{-\infty}^{\infty} \nu^n \sin \omega_0 \nu \, d\nu \\
&= \frac{2n!}{\omega_0(n+1)} \begin{cases} (j^n) \cos \omega_0 t & n \text{ even} \\ (j^{n+1}) \sin \omega_0 t & n \text{ odd} \end{cases} \qquad (53)
\end{aligned}
$$

and

$$
\begin{aligned}
1/\pi \int_{-\infty}^{\infty} & (\sigma - t)^n \sin \omega_0 \sigma \, d\sigma \\
&= \frac{\sin \omega_0 t}{\pi} \int_{-\infty}^{\infty} \nu^n \cos \omega_0 \nu \, d\nu \\
&\quad + \frac{\cos \omega_0 t}{\pi} \int_{-\infty}^{\infty} \nu^n \sin \omega_0 \nu \, d\nu \\
&= \frac{2n!}{\omega_0^{n+1}} \begin{cases} j^n \sin \omega_0 t & n \text{ even} \\ j^{n-1} \cos \omega_0 t & n \text{ odd} \end{cases} \qquad (54)
\end{aligned}
$$

whence the remainder term, Q, in (52) is

$$
\begin{aligned}
Q/2 &= \frac{k' \cos \omega_0 t + J' \sin \omega_0 t}{\omega_0} \\
&\quad + \frac{-K'' \sin \omega_0 t + J'' \cos \omega_0 t}{2\omega_0^2} \\
&\quad + \frac{-K''' \cos \omega_0 t - J''' \sin \omega_0 t}{3\omega_0^3} \qquad (55) \\
&\quad + \cdots \text{ etc.}
\end{aligned}
$$

Now if ω_p is the highest significant frequency component in $K(t)$ or $J(t)$, then average magnitude of the mth derivative is at most $\omega_p{}^m$ times that of $K(t)$ or $J(t)$. From which

$$
\frac{Q}{2} = \frac{\omega_p}{\omega_0} \sigma \left[R(t) \begin{matrix} \sin \\ \cos \end{matrix} \theta(t) \right] + \frac{\omega_p^2}{\omega_0^2} \sigma \left[R(t) \begin{matrix} \sin \\ \cos \end{matrix} \theta(t) \right] + \cdots \qquad (56)
$$

where $\sigma[\quad]$ means "is at most of the order of" or

$$
f_c(t) = -R(t) \sin [\omega_0 t + \theta(t)] + 2\sigma\left[\frac{\omega_p}{\omega_0}\right] R(t) + \cdots. \qquad (57)
$$

Therefore, for a narrow-band signal $f(t)$, $f_c(t)$ has the same envelope but a carrier shifted in phase by 90°, with the error of the order of the percentage bandwidth.

Acknowledgment

The work reported here represents the joint efforts of many people. With the lapse of time it is difficult to construct a precise or complete list of credits. W. Siebert, J. Granlund, D. Gray, and later R. Manasse all contributed to the theoretical understanding of the problem. S. Autler and G. Pettengill helped in the development of the delay lines. L. G. Kraft, H. Penfield, J. Petrishan, E. Howlett, and R. Keyes all made substantial contributions in the design, construction, and testing of the experimental apparatus.

This development was carried out as a part of a project under the over-all supervision of W. Siebert. It is difficult to overestimate the value of Professor Siebert's original contributions to all phases of the work.

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Optical Data Processing and Filtering Systems*

L. J. CUTRONA†, E. N. LEITH†, C. J. PALERMO†, AND L. J. PORCELLO†

Summary—**Optical systems, which inherently possess two degrees of freedom rather than the single degree of freedom available in a single electronic channel, appear to offer some advantages over their electronic counterparts for certain applications. Coherent optical systems have the added property that one may easily obtain many successive two-dimensional Fourier transforms of any given light amplitude distribution, or, by use of astigmatic optics, one-dimensional transforms can be obtained. Therefore, most linear operations of an integral transform nature are easily implemented. The optical implementation of integral transforms which are of importance to communication theory is discussed; the general problems of optical filter synthesis and multichannel computation and data processing are introduced, followed by a discussion of potential applications. Astigmatic systems, which permit multichannel operation in lieu of two-dimensional processing, are treated as a special case of general two-dimensional processors. Complex input functions are discussed with relation to their role in coherent optical systems.**

A. Introduction

OPTICAL images have the inherent property that they possess two degrees of freedom, as represented by the two independent variables which define a point on a surface. In this respect, optical systems differ basically from electronic systems, which possess only time as an independent variable.

Optical systems have the additional property that a Fourier transform relation exists between the light amplitude distributions at the front and back focal planes of a lens used in such a system. This property may be put to use in coherent optical systems. An optical arrangement which presents a space-domain function and successive Fourier transforms is easily implemented. As a result, integral transform operations may often be carried out more conveniently in an optical system than in an equivalent electronic channel. Because of this Fourier transform relation, coherent optical systems behave, in many ways, analogously to electrical-filters. The ease of synthesis of these optical filters has recently made them useful in some areas where only electrical-filter networks were previously used.

In many problems arising in the field of communication engineering, a piece of electronic equipment is required to operate on an incoming signal so as to evaluate an integral of either of the general forms

$$I(x_0, y) = \int_{a(y)}^{b(y)} f(x, y)g(x - x_0, y)\, dx \qquad (1)$$

where x_0, a, and b may be functions of time, or

$$I(x_0, y_0) = \int_c^d \int_{a(y)}^{b(y)} f(x, y)g(x - x_0, y - y_0)\, dx\, dy \qquad (2)$$

where x_0, y_0, a, b, c, and d may be functions of time.

Processes such as those of cross-correlation, autocorrelation, convolution, spectral analysis, and antenna pattern analysis are special cases of the integrals (1) and (2) as are also various linear integral transforms. The integral (2), which includes a second integration over the y variable, allows the generation of two-dimensional transformations.

Electronic computation and evaluation systems for performing the above integrations exist, but they suffer from disadvantages inherent in systems possessing only one degree of freedom. In the electronic case, time is the only available independent variable. This is a severe restriction if either 1) the integral is to be evaluated for a large number of different values of the parameter y, or 2) an integration over y is also required. In such cases scanning, time-sharing, or time-sequencing procedures must be employed.

In an optical system, however, two independent variables are available. Thus, the optical system can readily handle the two-dimensional operation without resort to scanning. Alternatively, for a one-dimensional process with a varying parameter, the second dimension can be used to provide a number of independent computing channels for various incremental values of the unused variable. In the optical system, the number of independent one-dimensional channels is limited only by the number of positions which can be resolved across the system aperture. The two-dimensional nature of an optical channel may therefore be exploited either to provide a true two-dimensional processor or to provide a multichannel filter bank. For certain types of operations, the second degree of freedom may permit a considerable equipment simplification and, partly for this reason, interest in optical processing has been growing for the past several years.

The basic optical theory which permits a filter-theoreti description of an optical data-handling channel is by n means new. The Fourier transform relations upon whic the spatial filtering is based are essentially the relation established by Huygens, Fresnel, and Kirchhoff, whil the spatial filtering is essentially that proposed by Abbe in his theory of image formulation in the microscop

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e.g., discussion in Born and Wolf[1]). In the past decade, ptical systems have been extensively discussed in terms f communication theory.[2-9] The present authors have nade use of astigmatic optical systems to achieve multi-hannel operation, as described in Section I, E, and have ynthesized complex filter functions of two variables. In ddition, bipolar and complex signals have been recorded s transparencies with positive transmittance, by use of a arrier frequency, and then optically converted back into ipolar or complex form by appropriate spatial filtering.

This paper outlines some of the fundamental principles nd techniques useful in understanding and designing oherent optical systems and indicates areas of potential pplications.

B. Elementary Optical Systems

Consider a piece of film having a transmittance function T; in general, $T = T(x, y)$, a function of the two variables which define the film plane P_1. If this transparency is illuminated with light of intensity I_0 (Fig. 1), the emergent intensity distribution is

$$I(x, y) = I_0 T(x, y).$$

Suppose that a second transparency is overlaid on the first, or that the first transparency is imaged on the second (with unity magnification, for the sake of simplicity). The intensity distribution of the beam which is emergent from the second transparency is then

$$I(x, y) = I_0 T_1(x, y) T_2(x, y),$$

where T_1 and T_2 are the transmittance functions of planes P_1 and P_2.

Integration over a plane could be accomplished by imaging the region of integration onto a detector whose resolution elemental size is greater than the image of the region. The detector might, for example, be a photocell or a small region of photographic film. Fig. 2 shows a simple optical system that evaluates the integral

$$I = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y) g(x, y)\, dx\, dy. \tag{3}$$

[1] M. Born and E. Wolf, "Principles of Optics," Pergamon Press, New York, N. Y.; 1959.

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Alternatively, the integration is accomplished if the detector samples equally the light emerging from all parts of the plane of integration. The second technique is in general the more practical.

In the event that integration in only one dimension is required and the remaining dimension is to be used to achieve multichannel operation, a method must be found to limit the integration to only one dimension. Astigmatic optical systems, in which the focal properties are different for the two dimensions, can achieve this result. The integral evaluated is then of the form

$$I(y) = \int_{a(y)}^{b(y)} f(x, y) g(x, y)\, dx. \tag{4}$$

The astigmatic optical system shown in Fig. 3 differs from that of Fig. 2 only through the presence of the cylindrical lens, which counteracts, for the y dimension only, the inherent tendency of lens L_2 to integrate the light emerging from the plane P_2.

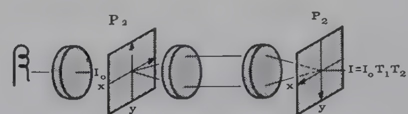


Fig. 1—Optical multiplication.

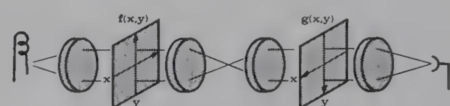


Fig. 2—Optical system for two-dimensional integration.

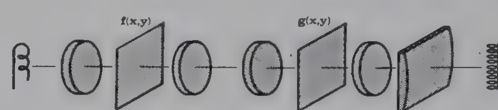


Fig. 3—Optical system for one-dimensional integration.

The more general case of evaluating

$$I(y, x_0) = \int_{a(y)}^{b(y)} f(x, y) g(x - x_0, y)\, dx \tag{5}$$

is easily handled by merely transporting the g transparency across the aperture in the x dimension. This is achieved by recording $g(x)$ on a strip of film which is then translated across the optical aperture, while all other elements remain stationary.

The optical systems discussed thus far impose the constraint that f and g must be everywhere positive. This arises because the transmission functions are merely energy ratios and are therefore always positive numbers,

$$0 \leq T(x, y) \leq 1 \quad \text{if} \quad T = \frac{I(x, y)}{I_0}.$$

In general, the f's and g's of interest are bipolar in nature. The representation of these f's and g's requires the use of a scaling factor k and a bias level B. Then $T(x, y) = B +$

$kf(x, y)$, where B and k are chosen such that $0 \leq T \leq 1$ for all x, y, and yet the term kf must not be negligible compared to B.[10] If one tries to evaluate an integral of the form (1) the integral generated is of the form

$$I(y, t) = \int_{a(y,t)}^{b(y,t)} [B_1 + k_1 f(x, y)][B_2 + k_2 g(x, y)]\, dx. \quad (6)$$

Error terms arise because of the presence of the bias levels. Even though the B's and k's are known, the removal of these errors or their exact calculation depends on the nature of f and g. The use of a "coherent" optical system will often eliminate certain of these difficulties which arise in the elementary systems described above.

C. Coherent Optical Systems

A coherent optical system is one in which the relative phases of the light waves in various parts of the system are invariant with time. For coherence to be achieved, a point source of light is required. Any two points in an optical system utilizing a point source have a relative phase which is time invariant.

The analysis of an optical system can in principle always proceed from the field equations. However, if only physical dimensions that are much greater than the light wavelength are considered, the analysis is simplified to the application of Huygens' principle. In this paper, the signals take the form of transparencies, the detail of which is sufficiently coarse that Huygens' principle can be applied.

A monochromatic electromagnetic wave is described by giving its magnitude and phase as a function of the three space variables. For example,

$$E_x = A(x, y, z) \cos [\omega t + \phi(x, y, z)],$$

where A is an amplitude factor, ϕ is a phase factor, and ω is the radian frequency of the wave, represents one component of the electric field vector. Since polarization effects are not of interest, any field vector can be denoted by E. Since only a few values of Z (where the optic axis is taken in the Z direction) are of concern, the field at z_1 is denoted by E_1, rather than by $E(Z)$.

Coherent optical systems have three fundamental properties which allow these systems to be analyzed by Fourier methods. Before the analysis is begun, two conventions will be adopted which will simplify the later discussions.

The first convention is that a wave of the form

$$E_1 = A(x, y) \cos [\omega t + \phi(x, y)]$$

will be written as

$$\hat{E}_1 = A(x, y) e^{j\phi(x,y)}.$$

[10] If $kf \ll B$, the kf term may become obscured by grain noise, stray light and so forth. On the other hand, if kf is fairly large, clipping may occur on signal peaks. An optimization process which determines the best choice of B and k for a particular set of signal and noise statistics and tolerable distortions is therefore necessary in some cases.

(whenever a letter appears with a "^" it is a complex fun tion.) The motivation for adopting this representation that all the significant features of the optical system a time invariant and that ω, the temporal radian frequenc acts, in a sense, like a carrier frequency.

The second convention relates to the description transparencies. Consider a thin transparency describe by a transmission function $t^2(x, y)$, and by a thickne function $\alpha(x, y)/2\pi(n - 1)$ (in wavelengths). Note th $0 \leq t \leq 1$, and n is the index of refraction of the tran parency. One can say that this transparency represen a signal function, $S(x, y)$, given by

$$S(x, y) = t(x, y) e^{j\alpha(x,y)}.$$

One derives the first property by noting that a lig wave,

$$E_1 = A(x, y) \cos [\omega t + \phi(x, y)],$$

incident on a transparency with transmission t^2, an thickness $\alpha/2\pi\ (n - 1)$ yields an emergent wave

$$E_2 = A(x, y) t(x, y) \cos [\omega t + \phi(x, y) + \alpha(x, y)],$$

or,

$$\hat{E}_2 = \hat{S}\hat{E}_1\,. \quad (7$$

The second basic property is related to the energy the wave. The energy ξ of a light wave E is proportion to the time average of E^2. Thus, the second property coherent optical systems is

$$\xi = k\hat{E}\hat{E}^* \quad (8$$

where * denotes conjugate. Since the only possible outpu of optical systems are in the form of energy-sensiti detectors (films, photoelectric cells, etc.), the comple representation gains added significance. In fact, t complex representation is sufficient for analyzing optic systems as "black boxes."

Consider the optical system shown in Fig. 4. Th

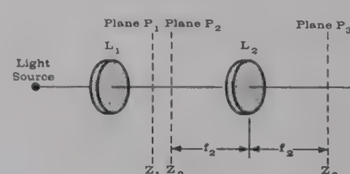


Fig. 4—Fourier transforms in an optical system.

following statements, which are demonstrated in Ap pendix I, summarize the third and most significa property of the complex representation. If the lig waves at planes P_1, P_2, and P_3 are denoted by $\hat{E}_1\ (x_1, y_1)$ $\hat{E}_2\ (x_2, y_2)$ and $\hat{E}_3\ (x_3, y_3)$, respectively, then $\hat{E}_3$ and form a Fourier pair to within a phase factor $e^{j\beta(x_3,y_3)}$, or

$$\hat{E}_3(x_3, y_3) = \mathfrak{F}[\hat{E}_1(x_1, y_1)] \cdot e^{j\beta(x_3,y_3)} \quad (9$$

where $\mathfrak{F}$ denotes the Fourier transform. Plane P_1 anywhere between L_1 and L_2, and β is a function of Z

n exact Fourier transform exists between $\hat{E}_3$ and $\hat{E}_2$, or

$$\beta(x_3, y_3) = 0 \quad \text{for} \quad Z_1 = Z_2; \tag{9b}$$

n exact Fourier transform does not exist anywhere else with respect to lens L_2).

$$\beta(x_3, y_3) \neq 0 \quad \text{for} \quad Z_1 \neq Z_2, \tag{9c}$$

A comment is perhaps in order at this point to avoid onfusion in some of the optical diagrams which follow. In onventional Fourier transform theory, the transforation from the time domain (the analog of the spatial lomain) to the frequency domain requires the kernel unction $e^{-i\omega t}$, and the transformation from frequency o time employs the conjugate kernel $e^{i\omega t}$. A lens always ntroduces the kernel $\exp j(2\pi/\lambda f)\ (x_n x_{n+1} + y_n y_{n+1})$ in assing from plane P_n to plane P_{n+1}. Therefore, in an ptical system one takes only successive transforms rather han a transform followed by its inverse. The effect of ısing a kernel of the wrong polarity, however, is simply o reverse the coordinate system of the transformed unction. Therefore, by reversing the coordinate system f the appropriate planes of Fig. 5, it is possible, in effect, o take inverse transforms, and to make the optical system onsistent with the conventions of Fourier transform heory.

The selection of the appropriate labeling may be conidered with the aid of Fig. 5. Arbitrarily call P_1 a spatial lomain plane, and consider the direction of propagation f the light wave to be from left to right. Lens L_1 introduces he kernel function $\exp j(2\pi/\lambda f)\ (x_2 x_1 + y_2 y_1)$. By defining

$$\omega_x = \frac{-2\pi}{\lambda f} x_2$$

nd

$$\omega_y = \frac{-2\pi}{\lambda f} y_2$$

he kernel becomes $\exp - j(\omega_x x + \omega_y y)$, which is in accord vith convention.

Recalling that $\mathfrak{F}\{\mathfrak{F}[f(x)]\} = f(-x)$, one then observes hat x_1 is mapped into $-x_3$ as a result of the two successive ransforms. In other words, the image at P_3 is reversed. f the coordinate system of P_3 is reversed,

$$\hat{E}(P_3) = \mathfrak{F}^{-1}\{\hat{E}(P_2)\} = \mathfrak{F}^{-1}\{\mathfrak{F}[\hat{E}(P_1)]\}$$
$$= \hat{E}(P_1)$$

o within bandwidth limitations,

The resultant coordinate systems for successive planes re shown in Fig. 6. By adopting this scheme of labeling, ne can represent each frequency domain by the Fourier ransform of the spatial domain to its left, and each patial domain by the inverse Fourier transform of the requency plane to its left. Note, however, that this scheme f coordinate assignment is valid only for the case of the ight wave traveling from left to right.

D. Filter Synthesis

The properties of coherent optical systems thus presented permit the synthesis of a wide range of optical filters. Such an optical filter consists of a transparency inserted at some appropriate position in the optical system.

In the optical system of Fig. 7, a signal $s(x, y)$ is inserted at plane P_1. At plane P_2, the spectrum $S(\omega_x, \omega_y)$ is displayed. Suppose a transparency $R(\omega_x, \omega_y)$ to be inserted also at plane P_2. Such a transparency modifies the spectral content of the signal, effecting the operation

$$\hat{V}(\omega_x, \omega_y) = \hat{S}(\omega_x, \omega_y)\hat{R}(\omega_x, \omega_y).$$

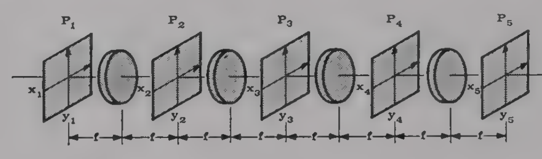


Fig. 5—Successive optical transforms.

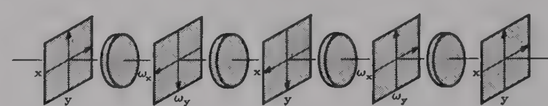


Fig. 6—Standard coordinates.

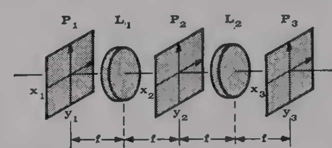


Fig. 7—The two-dimensional filter.

The transparency in general has a complex transmittance $\hat{R} = |R|\, e^{i\psi}$. The amplitude portion is obtained by varying the optical density and the phase portion is obtained by varying the thickness, which in turn varies the phase retardation.

At plane P_3, the signal is transformed back to the spatial domain, and is given by

$$\hat{v}(x, y) = \iint \hat{s}(x - \alpha, y - \beta)\hat{r}(\alpha, \beta)\, d\alpha\, d\beta,$$

where

$$\hat{v}(x, y) = \mathfrak{F}^{-1}[\hat{V}(\omega_x, \omega_y)], \tag{10}$$

and

$$\hat{r}(x, y) = \mathfrak{F}^{-1}[\hat{R}(\omega_x, \omega_y)].$$

This is a covolution integral.

The transparency $\hat{R}$ in its simplest form might be a slit or other aperture. Such apertures are low-pass or band-pass spatial filters. A stop becomes a band-rejection filter. The inclusion of a phase plate causes a phase shift of one portion of the spectrum with respect to the remainder. Complex filter functions are possible; because it appears

that one has independent control over both phase and amplitude, a wide variety of filter function can be synthesized.

As an alternative to inserting a transparency in the plane P_2 (the frequency domain), a transparency $\hat{r}(x, y)$, also with complex transmittance, can be introduced into the spatial domain at P_1. If provision is made for translating $\hat{s}(x, y)$ in the x-y plane relative to $\hat{r}(x, y)$, the signal at plane P_2 becomes $\hat{V}'(x', y') = \mathfrak{F}[\hat{s}(x - x', y - y')\, \hat{r}(x, y)]$. Here x' and y' measure the lateral displacement between s and r. At the position $\omega_x = \omega_y = 0$, the integral becomes

$$\hat{V}'(x', y') = \iint \hat{s}(x - x', y - y')\hat{r}(x, y)\, dx\, dy, \qquad (11)$$

which has the form of a cross-correlation. By reversing the coordinate system of $\hat{s}$ prior to recording, *i.e.*,

$$x - x' \rightarrow x' - x$$

$$y - y' \rightarrow y' - y,$$

one obtains the convolution integral

$$V'(x', y') = \iint s(x' - x, y' - y)r(x, y)\, dx\, dy.$$

This is identical in form with (10). Therefore, two methods are available for synthesizing a required transfer function:

1) The frequency domain synthesis, in which a complex transmittance function (called a filter) is introduced into the frequency domain, plane P_2, and operates directly on the frequency spectrum.

2) The spatial domain synthesis, in which a complex transmittance function (called a reference function) is introduced into the spatial domain, plane P_1, and operates directly upon the signal function.

The two techniques produce the same result, as indeed they should. The display is different, however. With the frequency domain operation of (10), an area display is produced in which the variables x, y are the coordinates of the plane P_3. With the spatial-domain operation, the output display is only a point (namely, $\omega_x = \omega_y = 0$) and the coordinates x', y' are generated as functions of time by physically displacing $\hat{s}$ with respect to $\hat{r}$. The spatial domain instrumentation requires a scanning mechanism; the filter technique does not.

It is possible and often advantageous to divide a required operation on $\hat{s}$ into two portions, one carried out in the spatial domain and the other in the frequency domain. The output at plane P_3 then becomes

$$\mathfrak{F}^{-1}\{\hat{R}_2(\omega_x, \omega_y)\mathfrak{F}[\hat{s}(x, y)\hat{r}_1(x, y)]\} \qquad (12)$$

where $\hat{r}_1$ is a reference function inserted in plane P_1 (the spatial-domain plane), and R_2 is a filter function inserted in plane P_2 (the frequency-domain plane).

A class of filters that has received considerable attention in communication theory is the matched filter, which has a transfer function that is the complex conjugate of the signal spectrum to which the filter is matched. Such a filter maximizes the ratio of the signal squared to the root-mean-square noise, when the noise is white Gaussian. The signals of concern here are two-dimensional and can also be complex. However, the above criterion for the matched filter still remains valid.

For the time-domain signals of electronics, the matched-filter criterion

$$\hat{R}(\omega) = \hat{S}^*(\omega)$$

implies for the impulse response the time-domain relationship

$$r(t) = s(-t).$$

For the optical case, since complex signals are possible, the corresponding relations are

$$\hat{R}(\omega_x, \omega_y) = \hat{S}^*(\omega_x, \omega_y)$$

and

$$\hat{r}(x, y) = \hat{s}^*(-x, -y)$$

where $\hat{r}$ and $\hat{s}^*$ are now complex conjugates.

The construction of the matched filter may be carried out by successive operations in the spatial and frequency domains, as suggested by (12). At plane P_1 of Fig. 7, a function $\hat{h}_1(x, y)$ is inserted. A filter $\hat{H}_2(\omega_x, \omega_y)$ is inserted at plane P_2 and the output is taken from plane P_3. The operation at P_1 produces

$$\hat{s}(x - x_1, y - y_1)\hat{h}_1(x, y),$$

and the filter $\hat{H}_2(\omega_x, \omega_y)$ produces, at plane P_3, the result

$$\iint \hat{s}(\alpha - x, \beta - y)\hat{h}_1(\alpha, \beta)\hat{h}_2(x_3 - \alpha, y_3 - \beta)\, d\alpha\, d\beta$$

where

$$\hat{h}_2(x, y) = \mathfrak{F}^{-1}[\hat{H}_2(\omega_x, \omega_y)].$$

If the observation in plane P_3 is confined to the position $x_3 = y_3 = 0$, the above expression reduces to

$$\iint \hat{s}(\alpha - x_1, \beta - y_1)\hat{h}_1(\alpha, \beta)\hat{h}_2(-\alpha_1 - \beta)\, d\alpha\, d\beta.$$

The matched filter condition is

$$\hat{h}_1(\alpha, \beta)\hat{h}_2(-\alpha, -\beta) = \hat{s}^*(\alpha, \beta)$$

or

$$\hat{H}_2(\omega_x, \omega_y) = \mathfrak{F}\left[\frac{\hat{s}^*(-\alpha, -\beta)}{\hat{h}_1(-\alpha, -\beta)}\right].$$

In most cases, the signal introduced into the optical system will be written as a real function, *i.e.*, written as a density variation on photographic film. However, it is often advantageous to use complex transparencies (phase control) for the synthesis of a required transfer function. For such a transparency $\hat{r}(x, y)$, there are three cases of interest:

1) $\hat{r}(x, y)$ is a varying-density transparency with uniform phase retardation over the aperture.

2) $\hat{r}(x, y)$ has uniform density over the aperture, but with phase retardation, $\phi(x, y)$.

3) $r(x, y)$ has both varying density and varying phase retardation over the aperature. The transparency in each case has, respectively, the form

1) $\hat{r}(x, y) = A(x, y)$,

2) $\hat{r}(x, y) = Ae^{i\phi(x,y)}$, $A =$ Constant,

3) $\hat{r}(x, y) = A(x, y)\ e^{i\phi(x,y)}$.

In each case, $A(x, y)$ is a real number $0 \leq A(x, y) \leq 1$ and $\phi(x, y)$ is real.

Consider the complex plane of Fig. 8. All values of $r(x, y)$ may be represented by a set of points within or on the unit circle. The functions $r(x, y)$ of case 1) lie on the real line $0A$. The functions $r(x, y)$ of case 2) lie on some circle within the unit circle ABCDA, whereas the functions $r(x, y)$ of case 3) may occupy any set of points within or on the unit circle.

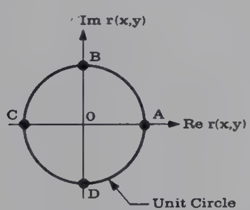


Fig. 8—Complex amplitude-transmission plane.

A special case of 2) is worth noting, *i.e.*, when $\phi(x, y)$ assumes only two values, differing by π (as, for example, 0 and π). This is a binary code. In terms of Fig. 8, the function is confined to two points lying on COA. Such a phase function is comparatively easy to generate, since techniques for producing phase gratings are well developed. By combining this with case 1), the entire line COA is utilized. Thus, real functions of positive and negative values can be used, as in the case of the electronic channel. In this case, however, no bias level is required.

The ability to process complex functions finds practical application in certain problems arising in radar and communication systems. In the most general case, one has a carrier signal which may be both amplitude- and phase-modulated. The modulated signal then takes the form

$$f(t) = A(t) \cos [\omega_c t + \alpha(t)],$$

which may be represented by the complex function

$$\hat{f}(t) = A(t)e^{i\alpha(t)}.$$

In a coherent system, one can make use of the phase function as well as of the amplitude function.

Generally, $\hat{f}(t)$ is to be subjected to some form of filter operation. For example, if $\hat{f}(t)$ represents a coded pulse, the operation of interest might be a cross correlation against a reference signal $\hat{s}(t)$. The operation can be performed optically if the complex functions $\hat{s}$ and $\hat{f}$ are appropriately recorded.

Within the present state of the art, it is difficult, however, to record simultaneously the modulus and argument of $\hat{f}$ in the form of a transmissivity and thickness variation. If the complex function $\hat{f}(t)$ is shifted in frequency by W (where W is greater than the highest significant frequency contributing to $Ae^{i\alpha}$), an alternative method becomes available. Only the real part of the frequency-translated $\hat{f}(t)$, *i.e.*,

$$\text{Re}\ \{A(t)e^{i[\alpha(t)+Wt]}\},$$

need be recorded, and this is displayed solely as a transmission variation. The frequency shift is effected by using a rotary phase shifter or equivalent device, while the real part is obtained through the use of a synchronous detector operating at the carrier frequency. Let $r(t)$ be the resulting function. The spectral display of the optical system is such that the positive and negative frequency components of a signal are independently available for attenuation and/or phase shifting. If the negative frequencies of $r(t)$ are then removed in an optical system, the resulting function is $A(t)\ e^{i[\alpha(t)+Wt]}$. (This is shown in Appendix II.)

Two observations deserve mention at this point. First, the presence of the radian frequency W should be taken into consideration in the optical filter. Second, the simplification in recording which is provided by the above technique exacts a price. One must double the bandwidth of the electronic channel at all points following the synchronous detector.

E. Astigmatic Optical Systems

The optical systems discussed thus far perform two-dimensional operations. This feature is useful if the signal to be processed is a function of two variables, because then the signal can be displayed as a two-dimensional function and processed simulataneously in both variables. An electronic system, having time as its only available independent variable, would require a scanning technique to perform the two-variable operation.

More often the signal is one-dimensional in nature and the additional variable is not required. In such a case, the second variable can be used to provide a multiplicity of independent channels so that many one-dimensional signals can be processed simultaneously. The signals to be processed are written as $\hat{s}_y(x)$, and are stacked on the transparency with respect to the y variable. The resultant transparency is of the form $\hat{s}(x, y)$ as before, except that now y is a parameter which takes on as many values as there are independent channels to process. The limit on this number is, of course, the number of resolvable elements available across the y dimension of the optical system aperture.

The processing is to be done with respect to the x variable only. The y-dimension channels must remain

separated. The optical system of Fig. 7, when modified as shown in Fig. 9, performs in the required manner.

The plane P_1, as before, has the coordinate system (x, y). At the Fourier transform plane P_2, a display of (ω_x, y) is desired; *i.e.*, one wishes to effect a Fourier transform with respect to x only, while preserving the y dimension. A cylindrical lens, which has focal power in one dimension only, can effect a one-dimensional Fourier transformation. To display the y dimension at P_2, plane P_1 is imaged at P_2; *i.e.*, a double Vourier transformation with respect to the y variable is instituted between P_1 and P_2.

A cylindrical lens $L_{cy(1)}$ in combination with a spherical lens L_2 is placed between planes P_1 and P_2. The cylindrical lens exerts focal power in the y dimension only and, by itself, produces a Fourier transformation with respect to y. The lens L_2, by itself, introduces a two-dimensional Fourier transformation. The two lenses in combination produce a double transformation with respect to y and a single transformation with respect to x. Plane P_2, therefore, has ω_x and y as its coordinates. Ideally, of course, one would like to transfer the y dimension of P_1 directly to plane P_2. This is possible only by imaging P_1 and P_2, which, of course, implies a double Fourier transformation.

At the plane P_2, a filter element $R(\omega_x, y)$ is inserted. This function is interpreted as a multichannel, one-dimensional filter which processes each channel independently. A similar cylindrical-spherical lens combination between planes P_2 and P_3 results in the inverse transformation with respect to ω_x. Thus, the output plane P_3 with coordinates x, y displays the input function after modification by the filter.

A simplification is possible if the signals in all channels are processed identically. Separation of the channels at the plane P_2, or frequency plane, is no longer necessary. In this event, the cylindrical lenses are not required and the optical system reverts to that of Fig. 6. The function displayed at the frequency plane P_2 is, as earlier, $S(\omega_x, \omega_y)$. The filter element, being independent of ω_y, takes the form $R(\omega_x)$. The function displayed at plane P_3 is modified with respect to the x dimension frequencies only. The operation can be written as

$$\hat{v}(x, y) = \mathfrak{F}^{-1}[\hat{S}(\omega_x, \omega_y)\hat{R}(\omega_x)]$$

$$= \int \hat{s}(x - \alpha, y)\hat{r}(\alpha)\, d\alpha.$$

As in the two-dimensional processor, the required transfer function can be synthesized in the spatial domain. The optical system of Fig. 9 suffices, except that the output is taken from plane P_2, and the portion of the system beyond this plane is not required (Fig. 10). The integral evaluated when a slit is placed along the line $\omega_x = 0$ is

$$\hat{V}'(x', y) = \int \hat{s}(x - x', y)\hat{r}(x, y)\, dx,$$

where, as before, y is a parameter providing multichannel operation.

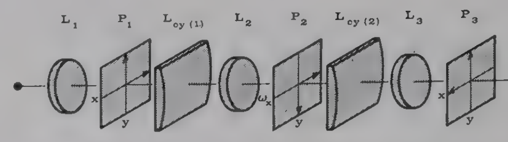


Fig. 9—Multichannel optical system.

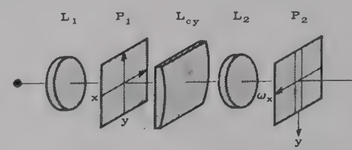


Fig. 10—Spatial domain filtering.

Instrumentations which require only one-dimensiona processing need be coherent in one dimension only Therefore, in the optical systems described in this section the point source of illumination can be replaced by a lin source oriented parallel to the y dimension. This is o practical advantage because the available light flux ca be increased by several orders of magnitude.

II. Potential Applications

The first part of this paper has been devoted to a exposition of the theory of coherent optical systems, in framework convenient for the purpose of filter design o the design of data-handling systems. Emphasis has bee placed on a filter-theoretic approach with no loss o generality, since all the integral-transform operation of interest in the field of communication theory can b described as filtering operations.

With the theory thus far presented, it is possible t discuss some potential applications of coherent optica systems. Innovations which are useful in particula problems would be incorporated into the simple optica systems already discussed. Amplitude and phase contro is available over two-dimensional regions in both th space and spatial-frequency domains; this permits grea flexibility of filter configurations.

However, many of the problems of interest at presen are inherently one-dimensional but have a varying param eter. If an optical configuration is sought, one is naturall led to an astigmatic arrangement. It is useful to conside an alternative approach to astigmatic optical systems they may be viewed as multichannel data-handling (o computing) systems. This alternative viewpoint is de veloped in Section II, A, and a discussion of possibl applications appears in Section II, B. The treatment o Section II, A will not involve any new theoretical concept and may be omitted in a first reading; however, it i intended to give some additional insight into the appl cation of astigmatic systems.

A. A Multichannel Optical Computer

The astigmatic systems of Section I, E can be instru mented as a multichannel optical computer with broa capabilities. The optical system can evaluate integra of the form

$$I(\omega_x, x_0, y) = \int_{a(y)}^{b(y)} f(x - x_0, y) g(x, y) e^{-i\omega_x x}\, dx. \qquad (13)$$

'he usefulness of an optical technique which evaluates ntegrals of this form will become evident when some otential applications are discussed in Section II, B. Iowever, the ease with which optical computations of he above form can be made warrants a few remarks at his point.

1) The functions $f(x, y)$ and $g(x, y)$ can be readily ecorded. The photographing of intensity-modulated athode-ray-tube presentations, or their equivalent, is a irect means for converting signal waveforms of very rge bandwidths to static form. While care is required 1 the design of the camera and film transport mechanisms hen multichannel operation is incorporated, the process essentially straightforward. Moreover, the staticizing f the signals means that signals with bandwidths of the rder of 100 mc per second can be recorded.

2) The Fourier transformation capability of the co- erent optical system permits simple sorting of spectra nd easy processing in the frequency domain.

3) The number of independent channels which can be andled simultaneously is limited only by the resolution f the photographic film and the resolving power of the nses. Generally, at least 20 channels per millimeter of lm can be accommodated.

The multichannel capability is readily demonstrated y placing an opaque obstacle with sides parallel to the axis in the plane P_1 of Fig. 9. This removes light from the orresponding interval in planes P_2 and P_3. Moving this bstacle in the y direction in P_1 (while keeping its edges arallel to the x axis) causes a corresponding image notion in planes P_2 and P_3. The edges of these corre- ponding image regions have a sharpness determined by he resolution of the optical system in the y direction.

Equipment in laboratory use employing commercially vailable components is capable of providing about 20 hannels per millimeter. Approximately 500 to 1000 hannels may therefore be obtained using 35-mm-size ptical components.

A further demonstration of the multichannel capability ollows. Let a transparency $A(x, y)$ be inserted into the lane P_1 of Fig. 9. This transparency consists of a number f grating strips with varying spatial frequency, the arious gratings being stacked in the y dimension. Each rating is in effect a square wave written about a bias evel (see Fig. 11).

In the plane P_2, a spectral analysis of $A(x, y)$, with espect to x will be found in each increment of y. The esult of photographing the light distribution in P_2 is iven in Fig. 12. It will be noted that all channels have a ecorded signal at the center. This corresponds, channel y channel, to the average level of illumination emergent rom that channel in plane P_1. The images in each strip how several lines on each side of the central image orresponding to the fundamental and harmonics com- rising the square wave in each y increment.

(a)

(b)

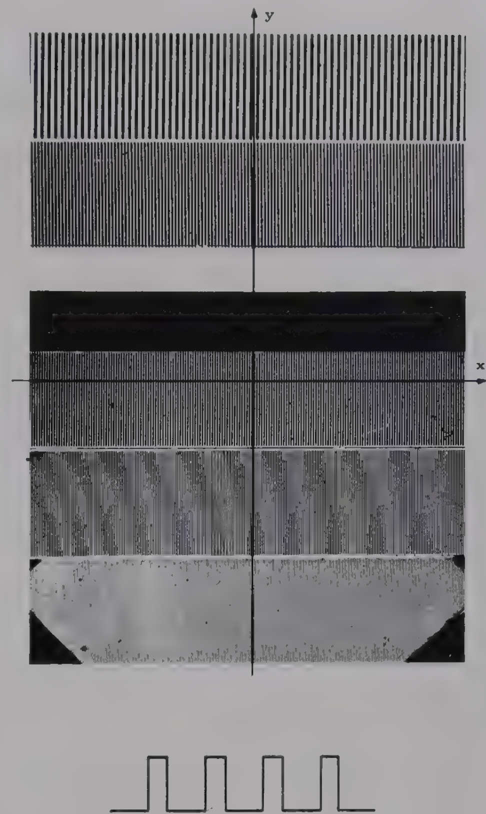


Fig. 11—(a) Multichannel diffraction grating. (b) Transmission function of grating for one channel.

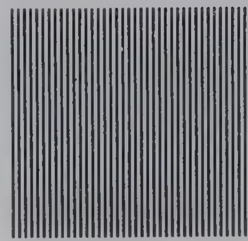
Fig. 12—Spectral analysis of the multichannel diffraction grating.

In some instances, for example, in taking a Fourier transform and making a frequency analysis, the output is taken from plane P_2. Alternatively, the spectrum at P_2 can be modified by means of stops, phase plates, or density filters, and the modified signal displayed at plane P_3. A photograph of the output at P_3 is shown in Fig. 13, for one channel only. If an obstacle is placed on the optical axis in plane P_2, the central image (*i.e.*, the dc or "bias" term) is removed and the contrast between the dark and light regions disappears, as shown in Fig. 13(b). This interesting behavior can be understood when one recalls that the eye (like any photographic recorder) responds to light intensity, the square of the light amplitude. Before removal of the bias level, the total light amplitude (bias plus square wave) was everywhere positive. Removal of the central image (the dc, or average, level) causes the amplitude of the light reaching P_3 to be bipolar; both positive and negative amplitudes appear. Since the eye and/or photographic recorders can only sense the square of this amplitude, positive and negative amplitudes cannot be distinguished from each other. Therefore, the contrast in P_3 vanishes when the dc is removed through filtering. Imperfect symmetry of the positive and negative half-cycles, and the loss of high-frequency components which accentuate the corners, tend to cause the residual effects. In particular, these are the dark lines between the half-cycles and the remaining slight contrast between positive and negative half-cycles.[11]

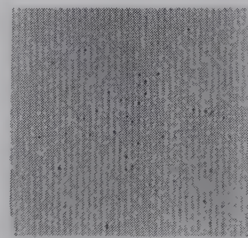
If the central image and first sideband on each side of the central image are allowed to pass plane P_2, another interesting feature may be demonstrated. If all higher-order images are removed at P_2, the image in plane P_3 resembles the pattern of P_1, except that the amplitude variation is now sinusoidal rather than square. The optical filter has passed the dc and the fundamental frequency, but rejected all harmonics. Fig. 14(a) shows the distribution in plane P_1, and Fig. 14(b) shows the resulting image in P_2. If, instead, the central image and the second sideband on each side (but not the first) were accepted, the image again would be sinusoidal, with twice as many lines per unit length present in P_3 as in P_1 [Fig. 14(c)]. This correspnds to selection of the average value and the second harmonic. Any frequencies present in the spectral decomposition of $A(x, y)$ (which is usually not completely symmetric in practice), may be selected and passed by the filter to form a resultant image in P_3, which is the transform of the spectrum in P_2.

In the process of evaluating an integral in the form of (13), it is often useful to perform a filtering operation on f or g before taking the product fg. The optical configuration of Fig. 9 is modified to that of Fig. 15 (opposite). Here, lens L_4 and cylindrical lens $L_{cy(3)}$ have been added. A transparency

[11] A useful operation closely related to the above discussion is performed in the phase-contrast microscope. Here, contrast is improved by retarding the dc component by a quarter-wavelength of the illumination. A quarter-wave plate placed on the axis of the system in the transform plane P_2 effects the retardation.

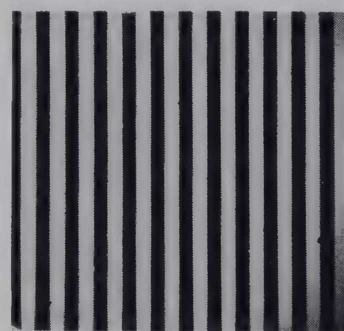


(a)

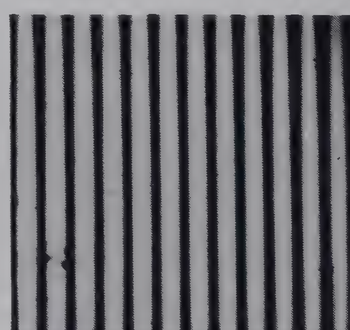


(b)

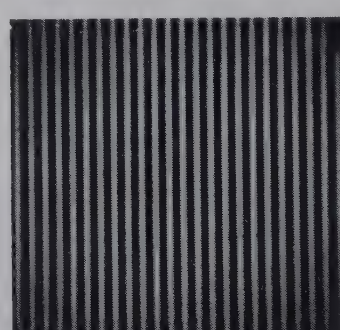
Fig. 13—(a) Square-wave diffraction grating. (b) Image of squar wave grating after bias removal.



(a)



(b)



(c)

Fig. 14—(a) Square-wave diffraction grating. (b) Image of squa wave grating when filter transmits dc and fundamental or (c) Image of square-wave grat.ng when filter transmits dc a second harmonic only.

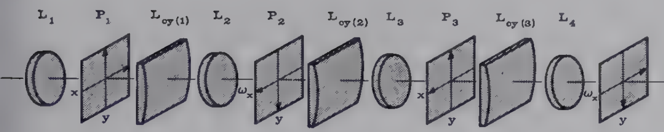


Fig. 15—Modified multichannel optical system.

$$A_1(x, y) = B_f + f(x, y)$$

s placed in P_1, and a second transparency

$$A_2(x, y) = B_g + g(x, y)$$

s placed in P_3, with coordinate axis properly chosen to orrect for image inversion effects. B_f and B_g are bias evels for the two transparencies.

In the absence of filtering in plane P_2, an image of P_1 ould be formed in P_3. This would illuminate $A_2(x, y)$ uch that the amplitude emergent to the right of P_3 would e the product

$$\begin{aligned} A_3(x, y) &= A_1(x, y)A_2(x, y) \\ &= [B_f + f(x, y)][B_g + g(x, y)] \end{aligned}$$

herefore,

$$\begin{aligned} A_3(x, y) = B_fB_g + B_gf(x, y) \\ + B_fg(x, y) + f(x, y)g(x, y) \qquad (14) \\ A_1(x, y). \end{aligned}$$

The presence of the bias terms results in three exraneous terms which in general interfere with the carrying ut of the desired operation. In particular, if a crossorrelation is to be performed, the output is taken from lane P_4 at the position $\omega_x = 0$. Unfortunately, the term B_fB_g, being a constant, puts energy into P_4 at $\omega_x = 0$. his seriously reduces the contrast in the output. Some dvantages of filtering in plane P_2 are now apparent. If n obstacle is place in P_2 at $x_2 = 0$, the dc component B_f f the function $A_1(x, y)$ may be removed. This makes the ght amplitude incident on plane P_3 proportional to $f(x, y)$ without the bias level. Phase information is present nd the bipolar function $f(x, y)$ has its positive and egative peaks 180° out of phase with each other. Thus, ne first and third terms of the right hand side of (14) anish.

A question now arises concerning the possible error in valuating (13) because of the second term of the rightand side of (14). The term $B_gf(x, y)$ can only be troubleome if $f(x, y)$, has a dc component (*i.e.*, if it contains patial frequency zero in the x direction). In this case, a omponent caused by this dc level will appear at $\omega_x = 0$ n plane P_4. The error caused by this term can, however, e removed from the system output by recording both $f(x, y)$ and $g(x, y)$ on a common carrier frequency. If the unctions resulting from writing $f(x, y)$ and $g(x, y)$ as a nodulation on a common carrier are designated as

$$A_1'(x, y) = B_f + f'(x, y)$$

and

$$A_2'(x, y) = B_g + g'(x, y),$$

then $B_gf'(x, y)$ will produce no output at $\omega_x = 0$ and it can be shown that

$$\begin{aligned} &\left[\int f'(x - x_0, y)g'(x, y)e^{j\omega_x x}\,dx\right]_{\omega_x=0} \\ &\qquad = \left[\int f(x - x_0, y)g(x, y)e^{j\omega_x x}\,dx\right]_{\omega_x=0}. \qquad (15) \end{aligned}$$

A proof of this equality is given in Appendix II.

B. Applications to Communication Theory

In preparing to discuss some possible applications of optical data processing techniques, it is useful to list some well-known mathematical operations which can be expressed in the form of (13). All the operations are taken as one-dimensional, but all except the Laplace transform permit a varying parameter. It is obvious that integration cannot be performed over an infinite interval, as some of the general expressions demand. However, in most practical cases $f(x, y)$ and $g(x, y)$ will vanish outside some interval. In Table I, $\pm L$ represents the achievable aperture limits of the optical system.

TABLE I

$I(\omega_x, x_0, y)$	$f(x - x_0, y)$	$g(x, y)$	$e^{j\omega_x x}$	$a(y)$	$b(y)$
Fourier transform	$f(x)$	1	$e^{-j\omega_x x}$	$-L$	$+L$
Laplace transform	$f(x - x_0)$	$g(x)$	$\omega_x = 0$	$-L$	$+L$
Cross-correlation	$f(x - x_0)$	$f(x)$	$\omega_x = 0$	$-L$	$+L$
Autocorrelation	$f(x_0 - x)$	$g(x)$	$\omega_x = 0$	$-L$	$+L$
Convolution	$f(x)$	e^{-xy}	$e^{-j\omega_x x}$	0	$+L$

A few realistic problems will now be discussed in terms of optical data processing and presentation.

1) *Fourier Transforms*: The Fourier transform provides the foundation for such areas as filter theory, spectral analysis, antenna-pattern analysis, and modulation theory, all of which are essential to the general field of electronics.

An optical spectrum analyzer is easily constructed to accommodate input signals in the form of a film record of a cathode-ray tube display. The technique of optically displaying the spectrum may be particularly useful in applications where one may wish to alter the spectrum in a prescribed manner, as in speech transmission and coding.

The study of antenna patterns also presents some interesting possibilities. It is well known that the far-field power pattern of an antenna can be determined from the squared modulus of the Fourier transform of the current distribution at the antenna aperture. In cases where one is interested in the effects of various illumination errors on the power pattern, the Fourier transform properties of coherent optical systems permit visual observation of the

resulting patterns as the optical aperture illumination is appropriately varied to correspond to antenna illumination errors.

One interesting mechanization is of particular interest to the filter designer. If a matched filter is to be designed for a signal of pulsed form with a constant repetition rate, the necessary filter is a comb filter. An electronic comb filter frequently is constructed using recirculating delay lines, a fairly complex mechanization. An optical comb filter, however, is merely a diffraction grating whose spacing, for a pulsed input signal, would be determined by the pulse repetition frequency of the signal.

The following experiments demonstrate some additional capabilities of optical processors.

First is a demonstration of optical modulation and demodulation. A transparency having, in one channel, an amplitude function of the form

$$A_1(x) = B_1 + \sigma_1 \cos \omega_c x$$

is multiplied by a second transparency

$$A_2(x) = B_2 + \sigma_2 \frac{\cos \omega_x}{|\cos \omega_x|}.$$

The first function $A_1(x)$ is the carrier, together with a bias term S_1, again required so that positive and negative values of the carrier can be represented as density variations on photographic film. The second funtion $A_2(x)$ is a square wave, along with a bias term S_2. For convenience in carrying out the experiment and in displaying the result, a relatively low carrier frequency has been chosen. The square wave function is shown in Fig. 16(a), and the carrier in Fig. 16(b). The experimental results are shown in Fig. 17.

The modulation process consists of a multiplication and a filtering operation. The former is carried out in the spatial domain. After the multiplication, frequencies are generated which are not part of the modulation function; these are the cross-product terms resulting from the bias S_1. Appropriately positioned stops act as stop-band filters and remove these components. The resulting function can be written as

$$B_1B_2 + B_2\sigma_1\left(1 + \frac{\sigma_2}{B_2} \frac{\cos \omega_x x}{|\cos \omega_x x|}\right) \cos \omega_c x$$

which is in the standard form of an amplitude-modulated carrier. Again, a bias S_1S_2 is required in order to be able to record the result. The modulated wave is shown in Fig. 16(c).

To demodulate the wave, two methods are available, each having an electronic analog. The first is a linear operation, somewhat analogous to synchronous demodulation. The spectral display at the frequency plane produces two spectral images for each frequency, symmetrically positioned about zero frequency. These can be regarded as produced from the exp $(j\omega_c t)$ and exp $(-j\omega_c t)$ portions of the function cos $\omega_c t$, *i.e.*, from the positive and negative frequencies. Suppose, now, that only half of the

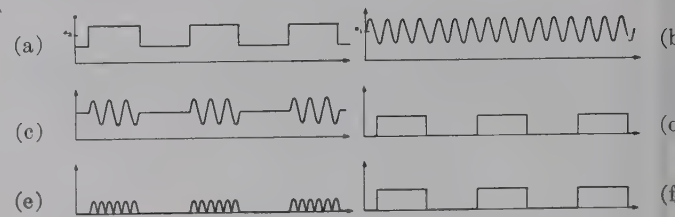


Fig. 16—(a) Original signal. (b) Carrier. (c) Modulated signa (d) Synchronous demodulation. (e) Full-wave rectification. (Full-wave rectification with filtering.

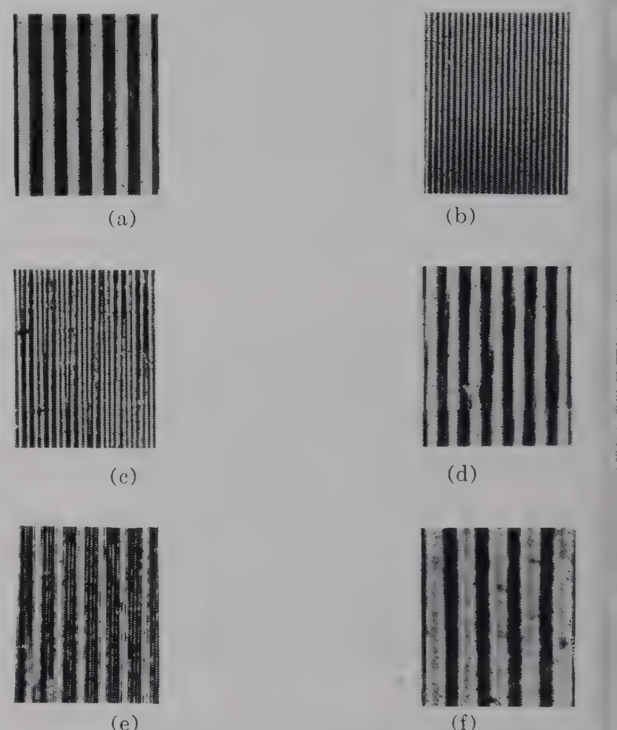


Fig. 17—Optical modulation and demodulation. (a) Original signa (b) Carrier. (c) Modulated signal. (d) Synchronous demodulatio (e) Full-wave rectification. (f) Full-wave rectification wi filtering.

frequency spectrum is transmitted (either the left right half plane). Also, let the zero frequency be eliminate The modulated wave becomes

$$B_2\sigma_1\left(1 + \frac{\sigma_2}{B_2} \frac{\cos \omega_x x}{|\cos \omega_x x|}\right) e^{j\omega_c x}.$$

A recording device, being insensitive to phase, records

$$\sigma_1\left(B_2\sigma_2 \frac{\cos \omega_x x}{|\cos \omega_x x|}\right),$$

as shown in Fig. 16(d). This is the original function befo modulation.

Alternatively, by removal of the bias term, the mod lated wave becomes full-wave rectified when record [Fig. 16(e)]. Additional filtering to remove the carr frequency terms produces the result in Fig. 16(f). T analogy of this operation to the detection operation electronics is quite apparent.

A second experiment demonstrates the use of indepe ent phase and amplitude control over portions of t

ectrum. The coherent optical system is used to obtain ie derivative of a function. This is done in the frequency omain. Differentiation of a spatial domain function rresponds to multiplication of the transform by ω. herefore, the appropriate filter for differentiation consts of the superposition of: 1) a transparency that is paque at the center or zero frequency region and that ecomes more transparent at higher frequencies; and 2) alf-wave phase plate needed to produce a reversal of gn for negative frequencies. Fig. 18 shows a pulse and s first two derivatives. The second derivative is obtained y use of a transparency with transmittance proportional) ω^2.

2) *Laplace Transforms*: The Laplace transform is given

$$F(s) = \int_0^\infty f(x)e^{-sx}\,dx$$

here s is a complex variable. Let $s = \alpha + j\beta$; then,

$$F(s) = F(\alpha, \beta) = \int_0^\infty f(x)e^{-\alpha x}e^{-j\beta x}\,dx,$$

hich has the form of a Fourier transform of the product f two functions (except for the lower limit on the integral). Iere β plays the role of a spatial frequency while one of he two functions contains a parameter α. The optical onfiguration to be used now follows from the discussion f astigmatic systems. Referring to Fig. 15, the planes P_1, P_2, P_3 and P_4 are, successively, Fourier transform airs with respect to the variable x. At plane P_1, thereore, one inserts an $f(x)$ which is uniform over all values f y. At plane P_3, P_1 is imaged onto the function $e^{-\alpha x}$ vhere the y dimension is used to provide the various alues of α. Finally, the function $F(\alpha, \beta)$ is displayed at lane P_4, where α and β correspond to the y and x direcions, respectively.

3) *Cross-Correlation*: The principles of optical crossorrelation have already been described; however, the implicity of the mechanization in certain cases is so xtreme as to deserve special mention.

Suppose $f(x)$ and $g(x)$ are two real-valued, narrow-band unctions which are to be cross-correlated. (The narrowand condition, bandwidth < center frequency, can be atisfied by proper modulation.) The following arrangeaent then yields the cross-correlation of $f(x)$ and $g(x)$:

1) The multichannel optical system of Fig. 9 is extended to have successive transform planes (in the x dimension) $P_1, \cdots, P_4$.
2) At P_1 the function $B_f + f(x)$ is placed in one channel. (For B_f sufficiently large, no thickness modulation is necessary, and the function is purely a varying-transmission transparency.)
3) Plane P_2 has a stop at zero frequency (on axis) which serves as a high-pass filter to reject B_f and accept $f(x)$.
4) At plane P_3, the resulting function $f(x)$ is imaged onto $B_g + g(x)$, thus forming the product $f(x)$ $[B_g + g(x + x_0)]$ where $x_0 = x_0(\tau)$, a displacement, may be a function of time.
5) Plane P_4 then contains the zero-frequency component of the integral of the above product on the system axis. Therefore, for each value $x_0(\tau)$ the output $h(\tau)$ is given by

$$h(\tau) = \int f(x)g[x + x_0(\tau)]\,dx.$$

It is simple to reject $B_g f$ because $f(x)$ is a narrow-band function.

One possible immediate application of optical cross-correlation as described above is in the reduction of data on atmospheric turbulence, where phase records are correlated to measure scale of turbulence, effective wind velocities, etc.

The extension of the technique to autocorrelation and convolution is obvious from the above discussion.

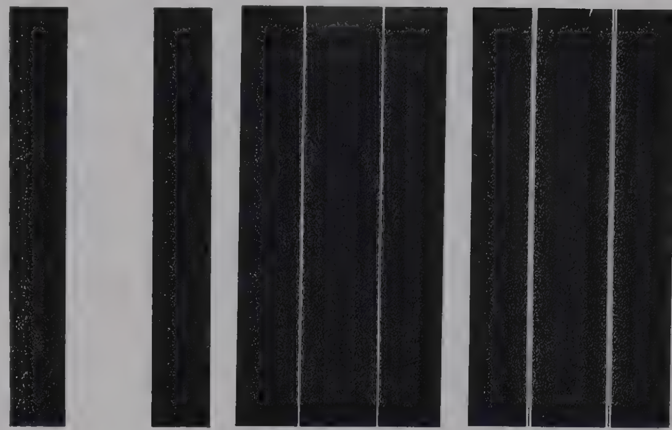
4) *Display of the Ambiguity Function*: The ambiguity function[12] is a useful tool in the analysis of radar and communication systems. Thus far there has been little success in the area of synthesis. Though one would like to specify the characteristics of the ambiguity function and then derive the signal function which yields these characteristics, the synthesis procedure is undeveloped, and one generally resorts to a trial and error method utilizing computer techniques.

As an example of the principles of coherent optical systems, three methods of displaying the ambiguity function will be given.

The ambiguity function $\hat{A}(t, \omega)$ associated with the signal function $\hat{f}(x)$ is given by

$$\hat{A}(\tau, \omega) = \int \hat{f}(x)\hat{f}^*(x + \tau)e^{-i\omega x}\,dx.$$

Generally one is concerned with the modulus squared of $A(t, \omega)$ rather than with $A(t, \omega)$ itself.



(a) (b) (c)

Fig. 18—Optical differentiation.

[12] P. M. Woodward, "Probability and Information Theory with Applications to Radar," McGraw-Hill Book Co., Inc., New York, N. Y.; 1953.

It has been shown that in an optical system there exist planes, P_1, P_2, $\cdots$, P_r, that are successive transform planes. These planes may be used either in a two-dimensional sense or in a multichannel sense. In order to make use of these concepts, one makes the following observations

1) The ambiguity function A is the Fourier transform of the product of two functions, $\hat{f}(x)$ and $\hat{f}^*(x + \tau)$;

or

2) The ambiguity function A is the cross-correlation of two functions, $\hat{f}^*(x)$ and $\hat{f}(x)\, e^{-i\omega x}$;

or

3) If $\hat{F}(\omega)$ is the Fourier transform of $\hat{f}(x)$, then $A(t, \omega)$ is the Fourier transform of the product $F(\xi - \omega)$ $F^*(\xi)$ (where * denotes conjugate).

Each of these observations leads to a different implementation.

1) In the first case, the function $f(x)$ is placed at plane P_1 [*i.e.*, for each value of y, the same function $f(x)$ is displayed]. The function $f(x + \tau)$ is placed at plane P_3 and the y dimension is used to get various values of τ. The product of $f(x)$ and $f(x + \tau)$ has now been completed by use of the multichannel capability. Finally, at plane P_4 the ambiguity function is displayed.
2) The function $f(x)\, e^{-i\omega x}$ is placed at plane P_1 and the y dimension is utilized for ω. The function $f(x + \tau)$ is passed through plane P_3, thereby utilizing real time for τ. The output is taken from a vertical slit placed on the optical axis at plane P_4. At any instant of time, $A(\omega, \tau_0)$ is displayed.
3) The third implementation is similar to the first except that the spectra $F(\omega - \omega_0)$ and $\overline{F(\omega)}$ are placed at planes P_1 and P_3, respectively, and the y dimension corresponds to ω.

III. Conclusions and Comments

The treatment of optical systems, primarily coherent optical systems, which has been presented in this paper shows that optical data processing and filtering systems may often present advantages over their electronic counterparts. The chief advantages stem from the following properties:

1) Optical systems are inherently two-dimensional.
2) Coherent systems inherently generate successive Fourier transform pairs.
3) Independent control over phase and amplitude of special components is easily effected.
4) Multiplication is effected by a simple imaging process.

Many other interesting properties then follow from these four.

However, many practical difficulties may arise to offset the possible advantages of a data-handling system. These, in general, originate from the use of photographic film as a medium for generating the function transparencies. The noise-like effects of film grain, perturbations in emulation thickness, and the effect of spurious scattering of light, all bear further investigation and, in some case present serious limitations. The question of comple independence between phase and amplitude control optical filters is treated in a manner which attributes ra properties to the light wave and therefore bypasses th question of field phenomena when very small objects a allowed to intercept the wave; a scattering analysis probably in order for this problem. Lens systems lim resolution and hence channel capacity and channel densit The delay time involved in developing photographic fil may be intolerable for some applications, Investigatio which will dictate the ultimate limitations and practic bility of optical computing or data-handling systems a far from complete. However, the flexibility and inhere implicity of optical channels appear to assure this tec nique a promising role in forthcoming filtering and dat handling problems.

Appendix I
Transform Relations in Coherent Systems

In Section I, C of this paper, the following three prope ties of coherent optical systems were given:

$$\hat{E}_3(x_3\,,\, y_3) = \mathfrak{F}\hat{E}_1(x, y_1)e^{j\beta(x_3, y_3)}, \quad (9$$

$$\beta(x_3\,,\, y_3) = 0 \quad \text{for} \quad Z_1 = Z_2\,, \quad (9$$

$$\beta(x_3\,,\, y_3) \neq 0 \quad \text{for} \quad Z_1 \neq Z_2\,. \quad (9$$

These properties are demonstrated with the aid of Fi 4. Suppose that lens L_1 forms a collimated beam of mon chromatic light. Calculation of $\hat{E}_3$ at x_3, y_3, given $\hat{E}$ requires finding the optical path length from x_1, y_1 x_3, y_3. $\hat{E}_3$ is then the integral, over plane P_1, of $\hat{E}_1$ proper delayed in phase according to the distance r,

$$\hat{E}_3 = \frac{1}{j\lambda}\iint \frac{1 + \cos\theta}{2\, d}\hat{E}_1(x_1\,,\, y_1)e^{-j(2\pi r/\lambda)}\, dx_1\, dy_1$$

where

λ = wavelength of light,
d = amplitude attenuation factor resulting fro distance between planes P_1 and P_3,
$\dfrac{1 + \cos\theta}{2}$ = the obliquity factor and
r = the distance between the points (x_1, y_1) a (x_3, y_3).

In the systems to be considered, $1/j\lambda$ is dropped becau absolute phase and amplitude are of no consequence, t d in the denominator is dropped because the distan attenuation is negligible, and the obliquity factor dropped because θ is always sufficiently small that c $\theta \approx 1$. The above expression becomes

$$\hat{E}_3 = \int \hat{E}_1 \exp\left[-j\frac{2\pi}{\lambda}r(x_1, y_1, x_3, y_3)\right] dx_1\, dy_1.$$

To calculate $r(x_1, y_1, x_3, y_3)$, consider the configurati of Fig. 19.

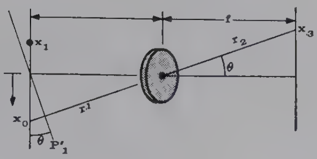


Fig. 19.

A plane wave P_1 making an angle θ with the normal s shown) is brought to a focus at x_3, where

$$x_3 = f \sin \theta.$$

his implies that the optical distance between x_3 and any oint on plane P_1' is a constant c. This constant is

$$= r_1 + r_2$$

$$= \sqrt{g^2 - x_0^2 \cos^2 \theta} + \sqrt{f^2 + x_3^2}$$

$$= g + f + \left(1 - \frac{g}{f}\right) \frac{x_3^2}{2f} \text{ for small } \theta, \text{ and using } \frac{x_0}{g} = \frac{x_3}{f}.$$

he distance from plane P_1 to x_3 is obtained by adding e term

$$-x_1 \sin \theta = \frac{x_1 x_3}{f}.$$

he total distance from x_1 to x_3 is

$$r(x_1, x_3) = g + f + \left(1 - \frac{g}{f}\right) \frac{x_3^2}{2f} - \frac{x_1 x_3}{f}$$

$$= A_0 + B_0 x_3^2 - \frac{x_1}{f} x_3.$$

his approach can be carried out in two dimensions to ield

$$(x_1, y_1, x_3, y_3) = \text{Const.} + \left[1 - \frac{g}{f}\right] \frac{(x_3^2 + y_3^2)}{2f} - \frac{x_1 x_3}{f} - \frac{y_1 y_3}{f}.$$

inally,

$$\hat{E}_3 = \left\{ \int \hat{E}_1 e^{-j\omega_x x_1} e^{-j\omega_y y_1} \, dx_1 \, dy_1 \right\} e^{j\beta(\omega_x, \omega_y)}$$

here

$$\omega_x = -\frac{2\pi x_3}{\lambda f},$$

$$\omega_y = -\frac{2\pi y_3}{\lambda f},$$

$$\beta = \left(1 - \frac{g}{f}\right)\left(\frac{x_3^2 + y_3^2}{2f}\right),$$

hich demonstrates (9a); (9b) follows immediately by etting $g = f$.

Condition (9b) can be obtained subject to weaker conitions than those required in the above analysis. It is ssumed only that the lenses are aberration-free, and ogether serve to image plane P_1 onto plane P_3 (Fig. 20).

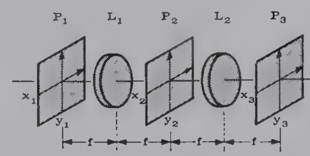


Fig. 20.

It is readily shown that each Fourier component $e^{-j(\omega_x x + \omega_y y)}$ at P_1 produces a plane wave which is brought to a focus on plane P_2 at position

$$x_2 = -f \frac{\omega_x \lambda}{2\pi}, \qquad y_2 = -f \frac{\omega_y \lambda}{2\pi}.$$

This implies that plane P_2 displays properly the two-dimensional power spectrum of the P_1 signal, but implies nothing about the relative phase of the spectral components. Thus, at plane P_2, one has

$$\hat{E}_2 = \mathfrak{F}[\hat{E}_1] \cdot \exp \{ j\beta_1(x_2, y_2) \}$$

where β_1 is the difference between the actual phase of $\hat{E}_2$ and the phase required by the transform relationship. Similarly,

$$\hat{E}_3 = \hat{E}_2 e^{j\beta_2(x_3, x_3)}.$$

If a real function is placed at plane P_1, *i.e.*,

$$S_1(x_1, y_1) = t(x_1, y_1),$$

then at plane P_3 an image of plane P_1 is observed, *i.e.*,

$$S_3(x_3, y_3) = S_1(-x_1, -y_1).$$

(The minus signs appear because successive Fourier transforms are taken with the kernel $e^{j\omega_x x}$.) One then has

$$t(-x, -y) = \mathfrak{F}\{ e^{j\beta_1} e^{j\beta_2} \mathfrak{F}[t(x- y)] \}$$

which implies

$$\beta_1 = \text{constant},$$

$$\beta_2 = \text{constant};$$

hence (9b) is proved to within a constant phase factor. Since phase is measured relative to the phase at $x_3 = y_3 = 0$, this constant is zero.

That only the planes P_1 and P_2 can be Fourier Transform planes is implied by the previous analysis. Again, however, this can be proved on the basis of a weaker assumption, namely that an aberration-free lens of arbitrary f number is used. Let a point source of illumination be placed in plane P_1. The Fourier transform of the point (a spatial impulse function) is a uniform amplitude function with linear phase shift.[13] Therefore, if P_1 and P_2 constitute a Fourier transform pair, the light at plane P_2 must be collimated. Therefore, P_1 must be in the focal plane of the lens. Similarly, the point source may be placed in plane P_2, in which case it is obvious that P_2 must be a focal plane for the transform relation to hold. Therefore, (9c) is proved.

[13] The linear phase shift is in fact a constant if the point source is on the optic axis.

Appendix II
Recording of Complex Functions

It is the purpose of this appendix to show how complex functions may be recorded as real functions and yet retain the essential features of the complex function. Eq. (15) of the text will be derived.

Suppose $\hat{f}(t)$ is a complex function having no frequencies higher than ω_0. Denote the Fourier transform of $\hat{f}(t)$ as $\hat{F}(\omega)$. Let $\hat{f}(t)$ modulate a carrier and call the resulting function $\hat{g}(t)$; *i.e.*,

$$\hat{g}(t) = \hat{f}(t)e^{i\omega_0 t}.$$

The spectrum of $\hat{g}(t)$, namely $\hat{\mathcal{C}}(\omega)$, is given by

$$\hat{\mathcal{C}}(\omega) = \hat{F}(\omega + \omega_0).$$

Instead of recording $\hat{f}(t)$, a complex function, Re $\hat{g}(t))$ is recorded

$$\text{Re}\ (\hat{g}(t)) = f_0(t) \cos\left[\alpha(t) + \omega_0 t\right]$$

if

$$\hat{f}(t) = f_0(t)e^{i\alpha(t)}.$$

Note that if $\hat{f}(t)$ *represents* an electronic signal, then Re $(g(t))$ can be obtained directly by first setting $\hat{f}(t)$ with a rotary phase shifter, thereby performing the operation,

$$f_0(t) \cos\left[\omega t + \alpha(t)\right] \rightarrow f_0(t) \cos\left[(\omega + \omega_0)t + \alpha(t)\right].$$

And then synchronously detecting the signal, produci

$$f_0(t) \cos\left[(\omega + \omega_0)t + \alpha(t)\right] \rightarrow f_0(t) \cos\left[\alpha(t) + \omega_0 t\right].$$

For positive frequencies, the spectrum of Re $(g(t))$ the same as that for $\hat{g}(t)$. (Recall that $\mathcal{C}(\omega) \equiv 0$ f $\omega < 0$.) Therefore, if the negative frequencies are reject optically, the resulting function is $\hat{g}(t)$ to within a consta multiplier.

Suppose one would like to perform the following ope ation on two complex functions $\hat{f}_1(t)$ and $\hat{f}_2(t)$:

$$\int f_1(t)f_2^*(t)e^{i\omega t}\,dt,$$

and suppose further that one would like *not* to reco complex functions. It has been shown above that one ca make available $\hat{f}_1(t)\ e^{i\omega_0 t}$ under the restriction of recor ing real functions. In a similar manner, $f_2^*(t)\ e^{-i\omega_0 t}$ ca be obtained and since the optics introduce the kern function $e^{i\omega t}$ automatically, the following operation carried out:

$$\int \hat{f}_1(t)e^{i\omega_0 t}\hat{f}_2^*(t)e^{-i\omega_0 t}e^{i\omega t}\,dt = \int \hat{f}_1(t)\hat{f}_2^*(t)e^{i\omega t}\,dt,$$

which is the desired operation. Eq. (15) of the text obtained by letting f_1 and f_2 be real.

Quaternary Codes for Pulsed Radar*

GEORGE R. WELTI†

Summary—A class of quaternary codes is described, and an algorithm for generating the codes is given. The codes have properties that make them useful for radar applications: 1) their autocorrelation consists of a single pulse, 2) their length can be any power of two, 3) each code can be paired with another code (its mate) of the same class in such a way that the crosscorrelation of mates is identically zero, 4) coded waveforms can be generated in a simple network the number of whose elements is proportional to the base-2 logarithm of the code length, and 5) the same network can be readily converted to a matched filter for the coded waveform.

I. Introduction

IT IS well known[1] that the *detection* capability of radar is independent of the transmitted sign waveform, if noise figure, antenna gain, and averag power are held constant, and if a matched filter is use in the receiver. It is also becoming widely recognized th the *range resolution* of a radar (*i.e.*, the minimum rang difference at which two equal point targets can be r

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[1] P. M. Woodward, "Probability and Information Theory, wi Applications to Radar," McGraw-Hill Book Co., Inc., New Yor N. Y.; 1953.

ved) is inversely proportional to the spectral bandlth of the transmitted waveform. Something that ·haps is recognized less widely, although it is often ›re important, is the fact that for a given integration ıe, the *range accuracy* (*i.e.*, the standard deviation of ıge error) is also inversely proportional to the bandlth.

Striving for fine resolution and accuracy has stimulated ›at interest in pulse-coding—a technique for increasing lar bandwidth without reducing average power or ›reasing peak power. In practice, the problem of design- ; a coded-pulse system usually entails a search for a itable rectangular-envelope waveform whose *ambiguity nction* is free of undesired maxima in the critical region ır the origin. Frequently, the figure-of-merit of interest the product of the duration and the spectral bandwidth the waveform, or the *time-bandwidth product*. A simple ıy to achieve a large time-bandwidth product is to vide the pulse into a sequence of contiguous subpulses d to choose at random the carrier phase in each sublse. Since the bandwidth of the resulting waveform (in cles per second) is the reciprocal of the subpulse dura- ın, the time-bandwidth product is equal to the number subpulses. The autocorrelation function of the waveım consists of a central spike embedded in a noise-like ckground, sometimes referred to as "hash." We shall fine *hash ratio* as the ratio of the peak magnitude of the sh to the magnitude of the central spike. A binaryled system has been described[2] in which the phase in ch subpulse is either 0 or π. In this system, the hash io can never be less than the reciprocal of the number subpulses. This bound results from the fact that for rtain arguments, the autocorrelation function is equal the bound multiplied by the sum of an odd number of egers, each having a value of either 1 or -1. The iter knows of no case in which the bound actually is ched when the number of subpulses exceeds 13.[3] The covery of longer subpulse sequences with acceptable sh ratios involves extensive search.

Receivers in coded-pulse systems contain a *matched er* whose impulse response is the time-inverse of the ›ected waveform of the received pulse. Matched filters linear by definition; they may be time-varying or ıe-invariant. Time-varying matched filters include the called correlation receivers. Strictly speaking, correon receivers are matched only at discrete times; hence, y are optimum only for targets at discrete ranges. This itation constitutes a weakness in some applications ., search systems); in others (*e.g.*, range-tracking tems) it does not. An advantage of correlation receivers hat the reference waveform can be generated using linear devices such as oscillators and flip-flops.

W. M. Siebert, "A radar detection philosophy," IRE Trans. Information Theory, vol. 2, pp. 204–221; September, 1956.
J. Storer and R. Turyn, "Optimum finite codes," *Proc. Symp. Recent Advances in Matched Filter Theory and Pulse Compression niques*, Rome Air Dev. Command, ARDC, USAF, Griffiss 3, N. Y., RADC-TR-59-161; September, 1959.

Great practical difficulties frequently arise when system considerations dictate the use of a time-invariant matched filter. The complexity of such a filter increases with increasing time-bandwidth product. For example, the number of cascaded delay lines or the number of sections in a lumped filter may become inordinately large.

We have pointed out two difficulties encountered in the design of coded-pulse radars with large time-bandwidth products. First, it is difficult to specify explicitly a satisfactory waveform. Second, it is difficult to construct a time-invariant matched filter. This paper proposes a quaternary coding scheme through which waveforms with arbitrarily large time-bandwidth products can be specified readily. Furthermore, it describes a simple canonic-form realization of the matched filter associated with the coded waveform.

Basic definitions are given in Section II. Section III describes a class of binary codes. This class is used as a basis for defining the quarternary codes discussed in Section IV. Section V treats the design and the properties of quaternary-coded waveforms. Matched filters are described in Section VI, and the paper is concluded in Section VII. An appendix contains proofs of theorems.

II. Basic Definitions

A *letter* is an element of an alphabet; its only property is that it can be distinguished from all other letters of that alphabet. An alphabet of length n is a set of n different letters.

We shall use the lower-case letters α, β, γ, ∂ of the Greek alphabet. We shall be dealing with two alphabets: a binary alphabet containing the letters α and β, and a quaternary alphabet containing the letters α, β, γ, ∂.

A *code* of length k is an ordered set of k *elements*, each of which is a letter of the same alphabet. We shall use the lower-case Roman letters with single subscripts to represent elements of a code, the numerical value of the subscript denoting the place of the element in the code.

A code may be designated in several ways—most simply by an upper case Roman letter (with or without subscripts and superscripts), more explicitly by showing expressions for several representative elements separated by commas and all enclosed in parentheses, or most explicitly by writing down the whole sequence of letters within parentheses, showing commas between symbols. Examples of code representations are the following:

$$A = (a_1 , \cdots , a_m),$$

$$C = (a_1 b_n , \cdots , a_n b_1),$$

$$F_7 = (\alpha, \alpha, \partial, \gamma).$$

The *inverse* of a given code is the code obtained by inverting the order of its elements. It is indicated by underscoring the code symbol. Thus, if

$$A = (\alpha, \beta, \gamma, \partial),$$

then

$$\underline{A} = (\partial, \gamma, \beta, \alpha).$$

The *product* of two *letters* is a scalar quantity that can be arbitrarily defined for every pair of letters. The product of two letters is indicated by writing the two letters next to one another; thus, the product of α and β is $\alpha\beta$. For binary alphabets, the product is obtained by some investigators by associating the scalar quantities 1 and -1 with the two letters of the alphabet and taking their arithmetic product. An equivalent procedure is to use the scalars 0 and 1 and to substitute modulo-2 addition for multiplication. Here, we shall find it convenient simply to define the product table and to avoid associating scalar quantities with letters. The following binary and quaternary product tables have been found useful and will be adhered to in this paper:

$$\begin{bmatrix} \alpha\alpha & \alpha\beta \\ \beta\alpha & \beta\beta \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha\alpha & \alpha\beta & \alpha\gamma & \alpha\partial \\ \beta\alpha & \beta\beta & \beta\gamma & \beta\partial \\ \gamma\alpha & \gamma\beta & \gamma\gamma & \gamma\partial \\ \partial\alpha & \partial\beta & \partial\gamma & \partial\partial \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Inspection of these matrices shows that the operation is commutative. No ternary product table (real or complex) could be found by the author with properties useful for the intended application.

The *product* of two *codes* corresponds to the scalar product of ordinary vectors and is defined by the formula

$$AB = \sum_{i=1}^{n} a_i b_i,$$

where

$$A = (a_1, \cdots, a_n),$$
$$B = (b_1, \cdots, b_n).$$

Codes will be called *orthogonal* if their product equals zero.

The *convolution* of two codes is a vector of scalar elements. This vector has properties analogous to those of an ordinary convolution function, inasmuch as each element of the vector is the sum of products of the elements of two "shifted" codes, one of which is inverted. Convolution is designated by placement of an asterisk between the two code symbols. Explicitly, we define

$$P = A*B = (p_{-n}, \cdots, p_n),$$

where

$$p_i = \begin{cases} (a_0, \cdots, a_{n+i})(b_{n+i}, \cdots, b_0), & i = -n, \cdots, -1 \\ (a_i, \cdots, a_n)(b_n, \cdots, b_i), & i = 0, \cdots, n \end{cases}$$
$$A = (a_0, \cdots, a_n),$$
$$B = (b_0, \cdots, b_n).$$

The *autocorrelation* of a code is defined as the c volution of the code and its inverse. Symbolically, autocorrelation of A is given by the expression A Similarly, the *crosscorrelation* of A and B is $A*B$.

The *composition* of two or more codes is the c obtained by writing the original codes one after anot Conversely, a code may be *partioned* into shorter co Notation is illustrated by the following formula, in wl C is the composition of A and B.

$$C = (A \vdots B) = (a_1, \cdots, a_m, b_1, \cdots, b_n).$$

We may also say that C is partitioned into A and B.

We shall need to define additional operations for bin alphabets and codes. The *sum* of *binary letters* is a bin letter and is defined as follows:

$$\begin{bmatrix} \alpha + \alpha & \alpha + \beta \\ \beta + \alpha & \beta + \beta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}.$$

The sum of binary codes is a binary code obtained adding corresponding elements; thus,

$$A + B = (a_1 + b_1, \cdots, a_n + b_n).$$

As defined here, addition corresponds to the modu addition of binary sequences defined elsewhere in literature.

Scalar multiplication of a *binary code* is defined onl the scalar is an integer. If k is an integer, we define

$$kA \doteq \begin{cases} (\alpha, \cdots, \alpha), & k \text{ even}, \\ A, & k \text{ odd}. \end{cases}$$

The operation is distributive; that is,

$$kA + kB = k(A + B),$$

and

$$kA + mA = (k + m)A.$$

The *intersection* of binary letters is an operation deno by the symbol $\cap$. It gives rise to a binary letter as defi below:

$$\begin{bmatrix} \alpha \cap \alpha & \alpha \cap \beta \\ \beta \cap \alpha & \beta \cap \beta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \alpha & \beta \end{bmatrix}.$$

The intersection of binary codes is defined as follov

$$A \cap B = (a_1 \cap b_1, \cdots, a_n \cap b_n).$$

[4] Our definition for the intersection of binary codes correspo to Reed's definition for the product of codes: I. S. Reed, "A c of multiple-error-correcting codes and the decoding scheme," I TRANS. ON INFORMATION THEORY, vol. 4, pp. 38–49; Septem 1954.

The *negative* of a binary code is the code obtained by interchange of the letters α and β, and is indicated by overbar. Thus, if

$$A = (\alpha, \alpha, \beta),$$

n

$$\bar{A} = (\beta, \beta, \alpha).$$

III. A Class of Binary Codes

Let A_i^k be a periodic binary code of length 2^k, where = 1, $\cdots$, k. The first 2^{i-1} places of A_i^k contain the letter the next 2^{i-1} places contain the letter β; the next 2^{i-1} ces contain the letter α, etc.

Let X_i^k be a vector of length k whose components are her 0 or 1. The subscript i is numerically equal to a ary number associated with the components of X_i^k; at is

$$i = \sum_{j=1}^{k} x_j^i 2^{j-1},$$

d

$$X_i^k = (x_1^i, \cdots, x_k^i).$$

Using the above definitions, we may now associate with ery vector X_i^k a code B_i^k by writing

$$B_i^k = \sum_{j=1}^{k} x_j^i A_j^k \ . \tag{1}$$

is defines a set of 2^k codes for each k. It is easily shown at

$$B_i^k B_j^k = \begin{cases} 0, & i \neq j, \\ 2^k, & i = j. \end{cases}$$

us, the codes of each such set are mutually orthogonal.[5] Additional distinct sets of mutually orthogonal binary des can be obtained by choosing any binary code of gth 2^k not contained in the B_i^k set and adding it to ch code in the set. For our purpose, a particularly eful choice is the code

$$C^k = \sum_{j=1}^{k-1} A_j^k \cap A_{j+1}^k \ . \tag{2}$$

ing this code, we define the codes

$$D_i^k = B_i^k + C^k. \tag{3}$$

r any k, the set of all D_i^k codes consists of 2^k mutually hogonal binary codes of length 2^k. The totality of these s for all positive k constitutes the class of binary codes present interest. We shall refer to codes of this class as *D-codes*. Moreover, we shall call k and i the *order* and *rank* of D_i^k, respectively. We shall call the vector X_i^k the *rank vector* of D_i^k.

The codes D_i^k and D_j^k are defined as mates if $|i - j| = 2^{k-1}$. We shall denote the mate of a D-code by a tilde. From previous definitions, we find that

$$\tilde{D}_i^k = D_i^k + A_k^k, \tag{4}$$

hence, that any code and its mate agree in the leading 2^{k-1} elements and disagree in the rest.

D_i^k and D_j^k are *neighbors* if $|i - j| = 1$ and max (i, j) is odd. A D-code and its neighbor agree in the odd elements and disagree in the even elements.

Theorem I: The composition of a D-code and its mate is a D-code of the next higher order and of the same rank:

$$D_i^k = (D_i^{k-1} \vdots \tilde{D}_i^{k-1}), \qquad i = 0, \cdots, 2^{k-1} - 1. \tag{5}$$

Corollary:

$$D_i^k = (D_{i-2^{k-1}}^{k-1} \vdots \bar{\tilde{D}}_{i-2^{k-1}}^{k-1}), \qquad i = 2^{k-1}, \cdots, 2^k - 1.$$

This theorem, together with the definition of mate, allows us to write down long D-codes without reference to the cumbersome definitions in terms of A_i^k, B_i^k, and C^k. If the rank of the code is less than half its length, the composition theorem applies, and we simply write down the code of the same rank and next lower order followed by its mate. If the rank of the code is not less than half its length, the corollary applies; therefore, we write down the code of the next lower order and of rank diminished by 2^{k-1} followed by the negative of its mate. Repeated use of these rules ultimately carries us to the first-order codes, which we can remember easily:

$$D_0^1 = (\alpha, \alpha)$$

$$D_1^1 = (\alpha, \beta).$$

At every stage of partitioning, we test the rank of the code to be partitioned to determine whether the theorem or the corollary applies.

The process of writing long D-codes is further simplified by the recognition that testing of rank at every stage of partitioning is equivalent to an examination of a particular component of X_i^k. More explicitly, if at some stage we wish to partition D_j^m, $m = 2, \cdots, k$, then we must use the theorem if $x_m^i = 0$, and we must use the corollary if $x_m^i = 1$, where x_m^i is the mth component of the vector X_i^k associated with the original code D_i^k. Moreover, we find that the first-order code called for as a result of the final stage of partitioning has a rank equal to x_1^i. We may therefore compose D_i^k directly by inspection of X_i^k without explicitly writing down the partition formulas first. For example, let us compose D_{12}^4. The number 12, expressed in four-place binary form, is 1100. These binary digits, in reverse order, are the components of X_{12}^4; hence, $X_{12}^4 = (0, 0, 1, 1)$. Since $x_1^{12} = 0$, the first two places of D_{12}^4 contain the elements of D_0^1, and we have $D_{12}^4 = (\alpha, \alpha, \cdots)$. Since

[5] The codes of any set correspond directly to the Reed codes onging to the set $\Phi_i{}^k$, for which $a_0 = 1$ (Reed notation). They closely related to the Walsh functions, discussed by: N. Wiener, onlinear Problems in Random Theory," Technology Press of ss. Inst. Tech. and John Wiley and Sons, Inc., New York, N. Y.; 21, 22; 1959.

$x_2^{12} = 0$, the next two elements of D_{12}^4 are the mate of (α, α); thus, we have $D_{12}^4 = (\alpha, \alpha, \alpha, \beta, \cdots)$. Since $x_3^{12} = 1$, the next four elements of D_{12}^4 are the negative of the mate of $(\alpha, \alpha, \alpha, \beta)$; hence, $D_{12}^4 = (\alpha, \alpha, \alpha, \beta, \beta, \beta, \alpha, \beta, \cdots)$. Finally, since $x_4^{12} = 1$, the last eight elements are the negative of the mate of the first eight, or

$$D_{12}^4 = (\alpha, \alpha, \alpha, \beta, \beta, \beta, \alpha, \beta, \beta, \beta, \beta, \alpha, \beta, \beta, \alpha, \beta).$$

We observe that we have an algorithm for finding D-codes: Given a D-code of some length, we can generate a D-code of twice that length by partitioning it in the middle thus,

$$D_i^k = (A \,\vdots\, B)$$

and then writing

$$D_i^{k+1} = (A \,\vdots\, B \,\vdots\, A \,\vdots\, \bar{B})$$

or

$$D_{i+2^k}^{k+1} = (A \,\vdots\, B \,\vdots\, \bar{A} \,\vdots\, B)$$

Theorem II: *The inverse of a D-code (or the negative of the inverse, depending on the rank and order of the code) is the mate of its neighbor*:

$$\left.\begin{array}{ll} n \ \text{odd:} & \underline{D}_i^k \\ n \ \text{even:} & \bar{\underline{D}}_i^k \end{array}\right\} = \begin{cases} \tilde{D}_{i-1}^k, & n \ \text{odd}, \\ \tilde{D}_{i+1}^k, & n \ \text{even}, \end{cases}$$

where

$$n = k + \sum_{j=1}^{k} x_j^i.$$

Corollary: Let d_m, d_n, d_p, d_q be four elements of a D_i^k-code placed symmetrically with respect to the center of the code, so that

$$m < n < p < q, \quad \text{with} \quad n - m = q - p$$

and

$$m + q = 2^k + 1.$$

Then

$$d_m d_n = \begin{cases} d_p d_q, & n - m \ \text{even}, \\ -d_p d_q, & n - m \ \text{odd}, \end{cases}$$

$$d_m d_p = \begin{cases} d_n d_q, & p - m \ \text{odd}, \\ -d_n d_q, & p - m \ \text{even}. \end{cases}$$

IV. A Class of Quaternary Codes

In Section III, we have defined a class of binary codes, the D-codes. We now define a related class of quaternary codes, the *E-codes*. The code E_i^k is obtained from D_i^k by replacing even-place α's and β's with γ's and ∂'s, respectively. Explicitly, if

$$D_i^k = (d_1, \cdots, d_m),$$

then

$$E_i^k = (e_i, \cdots, e_m),$$

where

$$e_i = \begin{cases} d_i, & i \ \text{odd}, \\ \gamma, & i \ \text{even}, \quad d_i = \alpha, \\ \partial, & i \ \text{even}, \quad d_i = \beta. \end{cases}$$

The notion of code pairs that are mates or neighbor extended from D-codes to E-codes by the follow definitions: the codes E_i^k and E_j^k are *mates* if, and only D_i^k and D_j^k are mates; they are *neighbors* if, and only D_i^k and D_j^k are neighbors.

In the following discussion, we shall make use of t special vectors, defined as follows: A pulse vector P^k i vector of $2^{k+1} - 1$ components whose 2^kth compon equals 2^k and whose other components equal zero. A *n vector* N^k is a vector of $2^{k+1} - 1$ components, all co ponents being zero. A code is a *pulse code* if its au correlation is a pulse vector. Two codes are *mutu exclusive* if their crosscorrelation is a null vector.

Theorem III: 1) *Every E-code is a pulse code*; 2) *E-code and its mate are mutually exclusive*:

$$E_i^k * \underline{E}_i^k = P^k;$$

$$E_i^k * \tilde{\underline{E}}_i^k = N^k.$$

The first part of Theorem III is the most import result in this paper. It is a direct consequence of quaternary multiplication table adopted in Section and it establishes a basis for waveform design, as c cussed in Section V. The process that led to the select of the multiplication table deserves some comment. T object was to find a table for which the first part Theorem III would hold for arbitrarily long codes. In search for a suitable table, certain necessary and suffici conditions were first established which, when satisfi would ensure that, given any pulse code, if such a co existed, a new pulse code could be found by twinning elements of the given code. Formation of a new code twinning is accomplished by the substitution of ordered pair of elements for every element in the origi code, using a fixed rule for substitution. A workable r for E-codes is

$$\alpha \to \alpha, \gamma$$

$$\beta \to \beta, \partial$$

$$\gamma \to \alpha, \partial$$

$$\partial \to \beta, \gamma.$$

It was found that no binary or ternary code could sati the conditions. This does not imply, of course, that binary or ternary pulse codes exist. Such pulse codes exist, but no twinning system exists that always yie new pulse codes from such pulse codes.

Next we examine the effect of a modified multiplicati table on the autocorrelation of E-codes:

$$\begin{bmatrix} \alpha\alpha & \alpha\beta & \alpha\gamma & \alpha\partial \\ \beta\alpha & \beta\beta & \beta\gamma & \beta\partial \\ \gamma\alpha & \gamma\beta & \gamma\gamma & \gamma\partial \\ \partial\alpha & \partial\beta & \partial\gamma & \partial\partial \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & -1 & \epsilon & -\epsilon \\ -1 & 1 & -\epsilon & \epsilon \\ \hline \nu & -\nu & 1 & -1 \\ -\nu & \nu & -1 & 1 \end{array}\right].$$

note that the original table is obtained if $\epsilon = \nu = 0$. reover, if $\epsilon = \nu = 1$, the E-code acquires the properties ts associated D-code. Let

$$M_i^k = E_i^k * \underline{E}_i^k = (m_{1-2^k}, \cdots, m_{2^k-1}),$$

ere

$$m_p = \sum_{s=p+1}^{2^k} e_s e_{s-p}, \qquad p = 1, \cdots, 2^k - 1.$$

even "shifts" (p even), only the 2×2 subtables in upper left and lower right corners of the multiplication e are involved. Therefore, from Theorem III,

$$m_p = \begin{cases} 2^k, & p = 0, \\ 0, & p \neq 0, \quad p \text{ even.} \end{cases}$$

odd shifts we obtain

$$= \nu \sum_{s=(p+1)/2}^{2^{k-1}} d_{2s}\, d_{2s-p} + \epsilon \sum_{s=(p+1)/2}^{2^{k-1}-1} d_{2s+1}\, d_{2s+1-p}, \qquad p \text{ odd.}$$

lication of the corollary of Theorem II enables us to ice the number of terms in each sum if $p < 2^{k-1}$. We ain

$$= \nu \begin{cases} \displaystyle\sum_{s=2^{k-2}+1}^{2^{k-2}+(p-1)/2} d_{2s}\, d_{2s-p} + \epsilon \sum_{s=2^{k-2}}^{2^{k-2}+(p-1)/2} d_{2s+1}\, d_{2s+1-p}, \\ \qquad p = 1, 3, \cdots, 2^{k-1} - 1, \\ \displaystyle\sum_{s=(p+1)/2}^{2^{k-1}} d_{2s}\, d_{2s-p} + \epsilon \sum_{s=(p+1)/2}^{2^{k-1}-1} d_{2s+1}\, d_{2s+1-p}, \\ \qquad p = 2^{k-1} + 1, 2^{k-1} + 3, \cdots, 2^k - 1. \end{cases}$$

similar derivation of m_p for negative odd p yields lar results, except for an interchange of ϵ and ν. This rchange results in a lack of symmetry in the auto- elation vector—a result that follows naturally from asymmetric multiplication table.

n upper bound for $|m_p|$, p odd and positive, can be blished through use of the fact that each term in the s equals either 1 or -1. The magnitude of the sum not exceed the number of terms:

$$m_p \leqq u_p$$

re

$$\begin{cases} \nu \dfrac{p-1}{2} + \epsilon \dfrac{p+1}{2}, & p = 1, 3, \cdots, 2^{k-1} - 1, \\ \nu\left(2^{k-1} - \dfrac{p-1}{2}\right) + \epsilon\left(2^{k-1} - \dfrac{p+1}{2}\right), & \\ \qquad p = 2^{k-1} + 1, 2^{k-1} + 3, \cdots, 2^k - 1. \end{cases}$$

The least upper bound for u_p is given by

$$\max(u_p) = \begin{cases} \nu(2^{k-2} - 1) + \epsilon 2^{k-2}, & \nu < \epsilon \\ \nu 2^{k-2} + \epsilon(2^{k-2} - 1), & \nu > \epsilon \end{cases}.$$

Direct evaluation of $|m_p|$, p odd, for particular E-codes reveals that this bound is not generally attained for $k > 3$. A representative graph of the autocorrelation of an E-code is shown in Fig. 1, with u_p indicated.

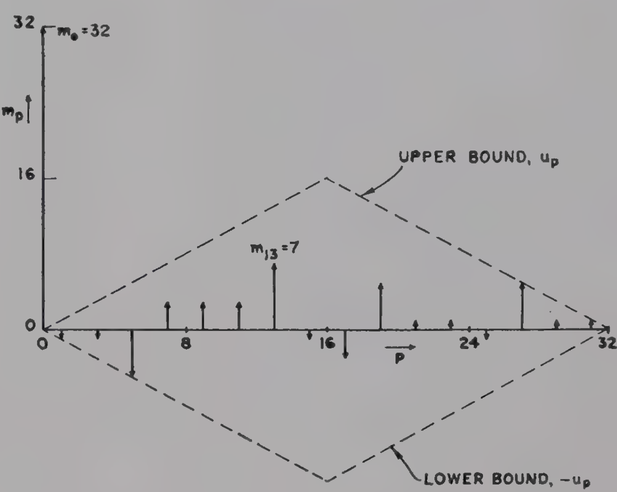

Fig. 1—Autocorrelation of $E_0{}^5$ with $\epsilon = \nu = 1$.

V. Quaternary-Coded Waveforms

Let the function $g(t)$ represent a waveform of finite duration, let $g(t) = 0$ when $|t| > \tau/2$, and let $h(t) = g(-t)$. We may synthesize functions of the form

$$f(t) = \sum_{i \text{ odd}} f_i g(t - i\tau + c) + \sum_{i \text{ even}} f_i h(t - i\tau + c),$$

where the coefficients f_i take on values from the set $(1, -1)$ and c is a constant. We shall call waveforms of the form $f(t)$ *quaternary-coded*, because they consist of a succession of subpulses of duration τ, each having one of the four waveforms $g(t)$, $-g(t)$, $g(-t)$, $-g(-t)$.

An *E-coded waveform* is defined as a quaternary-coded waveform for which

$$f_i = \begin{cases} e_i\alpha, & i \text{ odd,} \\ e_i\gamma, & i \text{ even.} \end{cases}$$

where the e_i are the elements of an E-code. If the E-code is of order k, we shall find it convenient to set $c = (2^k - 1)\tau/2$ in order to center the code with respect to the time origin.

Two E-coded waveforms are defined as *mates*, if, and only if, their associated E-codes are mates; they are *neighbors* if, and only if, their associated E-codes are neighbors.

An E-coded waveform consists of an alternation of an even number of $g(t)$ and $h(t)$ subpulses whose polarities are prescribed by the associated E-code. The leading and

trailing subpulses have the waveform $g(t)$ and $\pm h(t)$, respectively. The time-inverse of an E-coded waveform associated with a given E-code therefore consists of an alternation of $h(-t)$ and $g(-t)$ subpulses whose polarities are prescribed by the inverse of the E-code. Since the number of subpulses is even, and since $h(-t) = g(t)$, the inverse of an E-coded waveform (or its negative) is the E-coded waveform associated with the inverse of the given E-code. Therefore, from Theorem II, the inverse of a given E-coded waveform (or its negative) is the mate of the neighbor of the given waveform. We shall define two waveforms as *matched* if one (or its negative) is the time inverse of the other. The foregoing discussion constitutes a proof of the following Theorem:

Theorem IV: An E-coded waveform and the mate of its neighbor are matched.

We now examine the relationship between $E_i^k * E_i^k$ and the autocorrelation function of $f(t)$,

$$\phi_{ff}(t) = \int_{-\infty}^{\infty} f(u)f(u - t)\, du.$$

Substitution of the formula

$$f(t) = \sum_{j=1}^{2^k} f_j g_j\left(t - j\tau + \frac{2^k - 1}{2}\tau\right),$$

where

$$g_j(t) = \begin{cases} g(t), & j \text{ odd}, \\ h(t), & j \text{ even}, \end{cases}$$

and interchanging the orders of integration and summation yields

$$\phi_{ff}(n\tau + t) = \sum_{j=n+1}^{2^k} f_j f_{j-n} \phi_{j,j-n}(t), \quad n = 0, \cdots, 2^k - 1,$$

where $|t| < \tau/2$, and

$$\phi_{j,m}(t) = \int_{-\infty}^{\infty} g_j(s) g_m(s - t)\, ds.$$

Since

$$\phi_{j,m}(t) = \begin{cases} \phi_{11}(t), & j - m \text{ even}, \\ \phi_{12}(t), & j \text{ odd}, \quad m \text{ even}, \\ \phi_{21}(t), & j \text{ even}, \quad m \text{ odd}, \end{cases}$$

and

$$f_j f_m = e_j e_m,$$

we obtain

$$\phi_{ff}(n\tau + t) = \begin{cases} \phi_{11}(t) \displaystyle\sum_{j=n+1}^{2^k} e_j e_{j-n}, & n \text{ even}, \\ \phi_{12}(t) \displaystyle\sum_{\substack{j=n+1 \\ j \text{ odd}}}^{2^k} e_j e_{j-n} + \phi_{21}(t) \displaystyle\sum_{\substack{j=n+1 \\ j \text{ even}}}^{2^k} e_j e_{j-n}, & n \text{ odd}. \end{cases}$$

Comparison with the formulas for m_p in Section shows that for $t < \tau/2$,

$$\phi_{ff}(n\tau + t) = \begin{cases} 2^k \phi_{11}(t), & n = 0, \\ 0, & n \neq 0, \quad \text{even} \end{cases}$$

and

$$\phi_{ff}(n\tau + t) = \begin{cases} \phi_{12}(t) A_n + \phi_{21}(t) B_n, \\ \qquad n = 1, 3, \cdots, 2^{k-1} - \\ \phi_{12}(t) C_n + \phi_{21}(t) D_n, \\ \qquad n = 2^{k-1} + 1, 2^{k-1} + 3, \cdots, 2^k - \end{cases}$$

where

$$A_n = \sum_{s=2^{k-2}+1}^{2^{k-2}+(n-1)/2} d_{2s}\, d_{2s-n},$$

$$B_n = \sum_{s=2^{k-2}}^{2^{k-2}+(n-1)/2} d_{2s+1}\, d_{2s+1-n},$$

$$C_n = \sum_{s=(n-1)/2}^{2^{k-1}} d_{2s}\, d_{2s-n},$$

$$D_n = \sum_{s=(n+1)/2}^{2^{k-1}-1} d_{2s+1}\, d_{2s+1-n},$$

and where the d_p are the elements of the D-code associa with the E-code.

Combining equations now gives the following express for the autocorrelation function of an E-coded wavefo

$$\phi_{ff}(t) = 2^k \phi_{11}(t) + \sum_{\substack{n=1 \\ n \text{ odd}}}^{2^k-1} [E_n \phi_{12}(|t| - n\tau) + F_n \phi_{21}(|t| - n$$

where

$$E_n = \begin{cases} A_n, & n < 2^{k-1}, \\ C_n, & n > 2^{k-1}, \end{cases}$$

$$F_n = \begin{cases} B_n, & n < 2^{k-1}, \\ D_n, & n > 2^{k-1}. \end{cases}$$

Now, if the waveform $g(t)$ is suitably chosen, we ensure that

$$\max_t |\phi_{12}(t)| = \max_t |\phi_{21}(t)| \ll \phi_{11}(0),$$

so that the hash becomes small and

$$\phi_{ff}(t) = 2^k \phi_{11}(t),$$

approximately. We see that the shape of the autocorr tion function and, hence, the spectral bandwidth of waveform $f(t)$ are independent of k. On the other ha the time duration of the nonzero portion of the wavef is equal to $2^k\tau$. The time-bandwidth product, theref is 2^k times as great as the time-bandwidth produc $g(t)$. With a suitable waveform $g(t)$ we can produce wa forms of arbitrary duration having a fixed correla interval.

Let $f(t)$ and $\tilde{f}(t)$ be two distinct E-coded waveforms ose associated E-codes are mates. A derivation similar the derivation of $\phi_{ff}(t)$ shows that the crosscorrelation ction of $f(t)$ and its mate has a form similar to the ression for $\phi_{ff}(t)$, except for the omission of the term $_{11}(t)$. This property may prove useful in certain appli-ions. For instance, in an asynchronous communication tem using a binary channel, mating waveforms could transmitted. Mating matched filters used in the receiver ıld produce easily detected short pulses at their puts. Transmission of the same waveform a number times in succession would produce a train of readily olvable pulses at one of the receiver outputs.

The hash ratio, sidelobes, and ambiguity function of an oded waveform generally resemble those of the sub-se waveform $g(t)$. At distances greater than a subpulse gth away from the origin, the ambiguity function ng the time axis) is composed of the crosscorrelation ction of $g(t)$ and $h(t)$. For a linearly-swept FM wave-n $g(t)$, this crosscorrelation function is a linearly-pt waveform with the same range of frequencies but ce the duration.

VI. Canonic-Form Network

For certain radar applications, it is desirable to con-ict a time-invariant linear filter whose impulse response he coded-pulse waveform or the time inverse of that veform. The network shown in Fig. 2 fulfills this require-nt for E-coded waveforms. Its essential components a pair of input filters—matched filters with the impulse ponse $g(t)$ and $g(-t)$—and k delay lines. The desig-ions $E_i^j(t)$ and $\tilde{E}_i^j(t)$ denote locations where the impulse ponse of the network is an E-coded waveform associated h the code E_i^j and its mate, respectively. The delay of the jth stage has a delay of $2^{j-1}\tau$. Its output splits two branches, one of which has a polarity reversal. location of the negative branch is determined by the component of the rank vector X_i^k as follows: If $x_j^i = 0$, lower branch is negative; if $x_j^i = 1$, the upper branch egative. The two output signals of the jth stage are es. The validity of these assertions is readily verified tracing signals through the network and observing t, for $g(t) = \partial(t)$, where $\partial(t)$ is the Dirac delta function, ı stage of the network is a mechanization of the orithm (given in Section III) for generating a D-code. ing $g(t) \neq g(-t)$ mechanizes the rules used in trans-ning D-codes into E-coded waveforms.

Coded-pulse radars sometimes employ a pair of matched rs whose impulse responses are the time-inverses of another. The transmitted waveform is generated in filter after that filter is excited by a trigger pulse. other filter processes the return echoes. Since the filters need not operate simultaneously, it is possible onstruct both with the same physical elements. The ched filter for the filter of Fig. 2 is obtained by inter-nging the final outputs and by relocating the negative

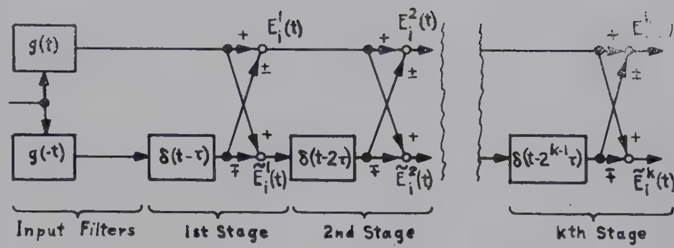

Fig. 2—Canonic-form filter with E-coded impulse response.

branch in the first stage of the network. That these changes achieve the required result can be inferred from Theorem IV. In the network of Fig. 2, the waveforms appearing at the two output taps are mates, and the neighbors of these waveforms are obtained by relocation of the negative branch in the first network stage.

A striking feature of the network of Fig. 2 is its economy of parts. The number of stages in the network increases as the base-2 logarithm of the time-bandwidth product of the coded waveform. In other words, a doubling of the time-bandwidth product is afforded by the addition of a single stage. Another feature of the network is its flexibility. The code can be made changeable by the insertion of suitable gates in the crossover sections to effect the necessary switching. This property may be useful in communication systems.

VII. Summary

An algorithm is given for generating a class of quaternary codes. Codes of this class have autocorrelation functions consisting of a single positive component embedded in an arbitrarily long sequence of zeros. It is shown that certain waveforms related to the quaternary codes have properties suitable for coded-pulse radars. Specifically, the time-bandwidth product of these waveforms can be made as great as desired, and their hash ratio can be strongly bounded. Another property of the waveforms is that each waveform can be paired with another waveform (its mate) with which its crosscorrelation is particularly low.

A canonic-form realization of the time-invariant matched filter for the waveforms is given. The canonic network consists of a cascade of lattice-like stages and has the property that the addition of a single stage doubles the time-bandwidth product of the network impulse response. Moreover, the same network can be used to produce coded waveforms, their mates, and their time-inverses.

Appendix

We shall find the following definitions useful: The code A^k is a code of length 2^k in which all elements are α's; the code B^k is a code of length 2^k in which all elements are β's.

Proof of Theorem I: From the definition of A_i^k, we have

$$\left(A_i^k \vdots A_i^k\right) = A_i^{k+1}. \tag{6}$$

From (1), we obtain

$$B_i^{k+1} = \sum_{j=1}^{k} x_j^i\left(A_j^k \vdots A_j^k\right) + x_{k+1}^i A_{k+1}^{k+1}.$$

But for $i = 0, \cdots, 2^k - 1$, the factor $x_{k+1}^i = 0$; hence

$$B_i^{k+1} = \left(B_i^k \vdots B_i^k\right), \qquad i = 0, \cdots, 2^k - 1. \tag{7}$$

From (2),

$$C^{k+1} = \sum_{j=1}^{k-1} A_j^{k+1} \cap A_{j+1}^{k+1} + A_k^{k+1} \cap A_{k+1}^{k+1}.$$

Use of (6) and of the identity $A_{k+1}^{k+1} = (A^k \vdots B^k)$ yields

$$\begin{aligned} C^{k+1} &= \sum_{j=1}^{k-1}\left(A_j^k \vdots A_j^k\right) \cap \left(A_{j+1}^k \vdots A_{j+1}^k\right) \\ &\qquad + \left(A_k^k \vdots A_k^k\right) \cap \left(A^k \vdots B^k\right) \\ &= \left(\sum_{j=1}^{k-1} A_j^k \cap A_{j+1}^k \vdots \sum_{j=1}^{k-1} A_j^k \cap A_{j+1}^k\right) \\ &\qquad + \left(A_k^k \cap A^k \vdots A_k^k \cap B^k\right) \\ &= \left(C^k \vdots C^k\right) + \left(A^k \vdots A_k^k\right). \end{aligned} \tag{8}$$

From (3), the addition of (7) and (8) gives

$$D_i^{k+1} = \left(D_i^k \vdots D_i^k\right) + \left(A^k \vdots A_k^k\right), \qquad i = 0, \cdots, 2^k - 1.$$

Since $D_i^k + A^k = D_i^k$, substitution of (4) results in

$$D_i^{k+1} = \left(D_i^k \vdots \tilde{D}_i^k\right), \qquad i = 0, \cdots, 2^k - 1$$

and the theorem is proved.

Proof of Theorem II: From the definition of A_j^k we find that $A_j^k + \underline{A}_j^k = B^k$. This identity can be used to expand the expression $\underline{A}_j^k \cap \underline{A}_{j+1}^k$

$$\begin{aligned} \underline{A}_j^k \cap \underline{A}_{j+1}^k &= (A_j^k + B^k) \cap (A_{j+1}^k + B^k) \\ &= (A_j^k + A_{j+1}^k) \cap B^k + A_j^k \cap A_{j+1}^k + B^k \cap B^k \\ &= A_j^k + A_{j+1}^k + A_j^k \cap A_{j+1}^k + B^k. \end{aligned} \tag{9}$$

Using (1) and (3), we write the expansion

$$\begin{aligned} D_i^k + \underline{D}_i^k &= \sum_{j=1}^{k} x_j^i(A_j^k + \underline{A}_j^k) \\ &\qquad + \sum_{j=1}^{k-1}\left[(A_j^k \cap A_{j+1}^k) + (\underline{A}_j^k \cap \underline{A}_{j+1}^k)\right]. \end{aligned}$$

Use of (9) gives the reduction

$$D_i^k + \underline{D}_i^k = \sum_{j=1}^{k} x_j^i B^k + \sum_{j=1}^{k-1}(A_j^k + A_{j+1}^k + B^k).$$

In the second summation, the terms A_j^k, $j = 2, \cdots, k - 1$ occur twice; hence, they make no contribution. We are left with

$$D_i^k + \underline{D}_i^k = \sum_{j=1}^{k} x_j^i B^k + A_1^k + A_k^k + \sum_{j=1}^{k-1} B^k.$$

The final form of the expression depends on whet B^k appears an even or an odd number of times:

$$D_i^k + D_i^k + A_1^k + A_k^k = \begin{cases} A^k, & n \text{ odd}, \\ B^k, & n \text{ even}, \end{cases}$$

where

$$n = k + \sum_{j=1}^{k} x_j^i.$$

This completes the proof.

Proof of Theorem III: We shall prove this theorem two steps. First, we shall show that if the theorem ho for E-codes of order $k - 1$, it must also hold for E-co of order k. Second, we show that it holds for E-codes order 1, and hence the theorem must be true. Theorer implies that

$$E = (L \vdots R) \text{ and } \tilde{E} = (L \vdots \bar{R}),$$

where L and R are E-codes and mates (or negati of mates). We may expand the autocorrelation of E follows:

$$\begin{aligned} E * \underline{E} &= \left(L * \underline{R} \vdots 0 \cdots 0\right) \\ &\qquad + \left(0 \cdots 0 \vdots L * \underline{L} \vdots 0 \cdots\right. \\ &\qquad + \left(0 \cdots 0 \vdots R * \underline{R} \vdots 0 \cdots\right. \\ &\qquad + \left(0 \cdots 0 \vdots R * \underline{L}\right). \end{aligned}$$

The crosscorrelation of E and $\tilde{E}$ is

$$\begin{aligned} E * \underline{\tilde{E}} &= \left(L * \underline{\bar{R}} \vdots 0 \cdots 0\right) \\ &\qquad + \left(0 \cdots 0 \vdots L * \underline{L} \vdots 0 \cdots\right. \\ &\qquad + \left(0 \cdots 0 \vdots R * \underline{\bar{R}} \vdots 0 \cdots\right. \\ &\qquad + \left(0 \cdots 0 \vdots R * \underline{L}\right). \end{aligned}$$

Now, if the theorem holds for L and R, these express reduce to $E * \underline{E} = P^k$ and $E * \underline{\tilde{E}} = N^k$, and the theo also holds for E and $\tilde{E}$. Let us test the theorem for the first-order codes (α, λ) and (α, δ):

$$(a, \lambda) * (\lambda, \alpha) = (0, 2, 0).$$

$$(a, \lambda) * (\delta, \alpha) = (0, 0, 0).$$

Since the theorem holds for first-order codes, it h for all codes, and our proof is complete.

;orrespondence

timum Approximation to a atched Filter Response*

n matched filter radar or communica- systems, the required matched filter ally is not realizable as a lumped param- network. Since the use of lumped ımeter networks is attractive, it is of rest to know how well one can do in roximating the matched filter by a ar system with only a few parameters t may be chosen freely. This note ribes an investigation of this approxi- ion problem and presents theoretical ılts on the use of simple RLC circuits ilters matched to a rectangular pulse.

et $g(t)$ be the impulse response of a r matched to the signal $g(-t) = 0$ for 0 and $t > T$. If $g(-t)$ is applied to a r whose impulse response is $g(t)$, the put will be maximum at $t = 0$. Now let be the impulse response of the approxi- ing filter, F. $h(t)$ is a function of a iber of parameters, $\alpha_i, i = 0, 1, 2, \ldots, n$. determined by the permitted complexity e filter, and the α_i are chosen according some criterion to give the optimum for a particular value of n. Two criteria might be used to determine the α_i are:

) Maximize (signal/rms noise)$^2_{\text{out}}$ at the time when the matched filter would have its maximum output, *viz.*,

$$\left[\int_0^T g(t)h(t)\,dt\right]^2 \Big/ \int_0^\infty h^2(t)\,dt$$

is to be a maximum.

) Minimize the integral of the square of the error between the desired and actual impulse response, *viz.*,

$$\int_0^\infty [g(t) - h(t)]^2\,dt$$

is to be minimized.

first criterion assumes white noise at r input and gives a result that is inde- lent of the gain of the filter. The second s a result that is gain dependent. How- , it has been proved that the two criteria to the same *form* of optimum impulse onse. Note, however, that in general maximum signal-to-noise ratio out of approximating filter, F, does not ssarily occur at the time when that of matched filter is maximum (at $t = 0$ is instance).

ie proof of the equivalence of the two ria is as follows: Maximization of al-to-noise ratio requires that

$$\frac{\displaystyle\int_0^\infty h(x)\frac{\partial h}{\partial \alpha_i}\,dx}{\displaystyle\int_0^T g(x)\frac{\partial h}{\partial \alpha_i}\,dx} = \frac{\displaystyle\int_0^\infty h^2(x)\,dx}{\displaystyle\int_0^T g(x)h(x)\,dx}, \qquad 0 \le i \le n.$$

Minimization of the error requires that

$$\frac{\displaystyle\int_0^\infty h(x)\frac{\partial h}{\partial \alpha_i}\,dx}{\displaystyle\int_0^T g(x)\frac{\partial h}{\partial \alpha_i}\,dx} = 1, \qquad 0 \le i \le n.$$

Let $h(t) = \alpha_0\eta(t)$, where α_0 is the optimum gain factor determined by the equation immediately above for $i = 0$, and $\eta(t)$ is a function of the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$. Both criteria then lead to the following condition:

$$\frac{\displaystyle\int_0^\infty \eta(x)\frac{\partial \eta}{\partial \alpha_i}\,dx}{\displaystyle\int_0^T g(x)\frac{\partial \eta}{\partial \alpha_i}\,dx} = \frac{\displaystyle\int_0^\infty \eta^2(x)\,dx}{\displaystyle\int_0^T g(x)\eta(x)\,dx}, \qquad 1 \le i \le n.$$

Therefore, $\eta(t)$ is the same for the two criteria.[1] Since the minimization of the time domain error

$$\int_0^\infty [g(t) - h(t)]^2\,dt$$

is equivalent to minimization of the frequency domain error

$$\int_{-\infty}^\infty |\,G(\omega) - H(\omega)\,|^2\,d\omega,$$

we have a third and equivalent criterion.

This result has been used to determine some optimum inpulse responses of the form $\eta_n(t) = e^{-\alpha_1 t}(1 + \alpha_2 t + \ldots + \alpha_n t^{n-1})$ to match a rectangular pulse of duration T. The transfer function of the resulting optimum filter has an nth-order real pole and $n - 1$ finite zeros. The results for $n = 1, 2, 3, 4$ are indicated in Table I (ρ is the SNR out of the optimum filter compared to that out of the matched filter) The results for $n = 2, 3, 4$ are new, and the result for $n = 1$ agrees with that given by Peterson and Birdsall.[2] For these cases it turns out that the output signal has its peak value at the same time as the matched filter, so within this class of networks, these are the best approximations to the matched filter. For $n = 2$ there is a value of $\alpha_1 T$ that gives a minimum value of ρ, and for $n = 3$ there is an $\alpha_1 T$ that gives a minimum and another that gives a secondary maximum value of ρ. For $n = 4$ there are two minima and two maxima for ρ. The parameters for the largest values of ρ are shown in the table. For $n = 2$, 3, and 4 there are more cases to be investigated (complex poles, separate poles on the negative real axis, or a combination of these) all of which require the solution of simultaneous transcendental equations.

TABLE I

n	$\alpha_1 T$	$\alpha_2 T$	$\alpha_3 T^2$	$\alpha_4 T^3$	ρ	$-10 \log \rho$
1	1.2564	—	—	—	0.8145	0.890 db
2	2.8427	10.476	—	—	0.8861	0.526
3	4.5376	−2.8841	34.990	—	0.9149	0.386
4	6.1807	13.830	−73.917	283.75	0.9304	0.313

As seen by the values of ρ in the above table, $n = 3$ is about the point of diminishing returns for the approximation when increase in complexity is weighed against increase in performance. A filter whose impulse response has an envelope of $\eta_3(t)$ has been constructed. Its impulse response, the pulse to which it is "matched," and its response to this pulse are shown in Fig. 1. Note the discontinuity of the slope of the envelope in Fig. 1(c) at the time corresponding to the end of the input pulse. The envelope in Fig. 1(a) is within 3 per cent of the designed envelope, and the frequency of the fine structure is within 1/2 per cent of the designed value. The measured value of ρ is within 0.1 db of the theoretical value of 0.915.

In order to determine how well a much more complex network approximates a matched filter for a rectangular pulse, the

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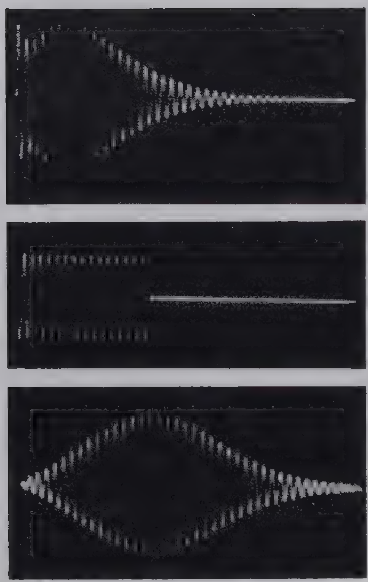
[1] George L. Turin of Hughes Research Laboratory has given (private communication) an elegant proof of the equivalence between maximizing the signal-to-noise ratio and an alternate form of error minimization, *viz.*, minimization of the integral of the square of the error between *normalized* forms of the desired and actual impulse responses. The normalization is with respect to energy. His proof uses a geometric visualization of these waveforms in signal-space.

[2] W. W. Peterson and P. T. Birdsall, "The Theory of Signal Detectability," Electronics Defense Group, Dept. of Elec. Engrg., Univ. of Michigan, Ann Arbor, Tech. Rept. no. 13; 1953.



(a)

(b)

(c)

Fig. 1—(a) Impulse response of optimum filter with a triple order pole and two finite zeros. (b) Pulse to which filter is matched. (c) Response of filter to (b).

SNR ratio at the output of a network designed by Strieby[3] was calculated. This network has 8 poles and 6 zeros, and its impulse response is an excellent (although not a least error) approximation to a rectangular pulse. $\rho = 0.96$ for this network.

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[3] M. Strieby, "Time Domain Synthesis by Means of Trigonometric Polynomial Approximations," Res. Lab. for Elec., Mass. Inst. Tech., Cambridge, Mass., Tech. Rept. no. 308; January, 1956.

Coherent Detection by Quasi-Orthogonal Square-Wave Pulse Functions*

With the rise of digital systems there has been an increasing need for pulse-communications techniques which operate efficiently in an extremely noisy environment. Statistical methods have been developed in an attempt to optimize desirable characteristics inherent in the transmission of information. Such procedures applied to digital transmission usually require that information be coded at the transmitter and coherently detected, using matched filters at the receiver. For good discrimination in the presence of noise it is sometimes desirable that the code symbols be orthogonal wide-band signals, having about the same energy per symbol,as per Bell[1] and Price and Green[2] (presuming, of course, that all symbols are equally likely).

It is well known that sine and cosine functions form an orthogonal set, and are therefore very useful in coherent detection of like signals. In this article the code symbols will be square-wave pulse functions $\phi_i(t)$ of differing repetition rates. A message will consist of a sequence of these functions equally spaced in time. When the square waves have equal amplitudes and time duration, the code symbols all have equal energies.

The remaining criterion governing the code symbols is that they be orthogonal. However, it is not really essential that the symbols be *strictly* orthogonal, but that the error due to nonorthogonality be small.

When, by proper choice, a set of functions is made to look orthogonal to any desired degree, the set of functions is said to be *quasi-orthogonal.* It will be shown in this paper that such a set may be composed entirely of square waves. When a signal coded by such waveforms is multiplied by a locally generated square wave of the same character, and integrated over one symbol length, the integral of the resultant is high; when the two square-wave signals are of differing character, the integral value is equal to, or smaller than, the orthogonal error allowed. Such a scheme could be used for those space-telemetry systems in which orthogonal codes have found advantage.[3]

Square-Wave Symbols

Let $\{\phi_i(t)\}$ be a set of (normalized) square-wave pulse functions, each $\phi_i(t)$ element of which is positive-going at $t = 0$, of unit magnitude, with n_i half cycles in the interval $(t = 0, 1)$, and zero elsewhere. Consider two elements, $\phi_i(t)$ and $\phi_j(t)$, of the set and form the integral

$$l_{ij} = \int_0^1 \phi_i(t)\phi_j(t)\,dt. \qquad (1)$$

Denote the highest common factor (greatest common divisor) of the pair n_i, n_j, as (n_i, n_j); similarly, denote their least common multiple as (n_i, n_j) (that is, the least number divisible by both n_i and n_j).

Theorem. The value of the integral in (1) above is

$$\int_0^1 \phi_i(t)\phi_j(t)\,dt = \begin{cases} 0 & \text{if } \dfrac{[n_i, n_j]}{(n_i, n_j)} \text{ is even} \\[2ex] \dfrac{(n_i, n_j)}{[n_i, n_j]} & \text{if } \dfrac{[n_i, n_j]}{(n_i, n_j)} \text{ is odd} \end{cases}. \qquad (2)$$

When $n_i = n_j$ the integral is unity; when n_i and n_j have unequal powers of 2 as factors, the integral is zero. Such functions, then, are othogonal. Stated another way, the set $\{n_i\}$ has elements $n_i = 2^{k(i)}n_i'$, where $k(i)$ is some nonnegative integ and n_i' is a positive odd integer.

Corollary. A set of square-wave pu of the type described is orthogonal and only if, all the elements of the $\{k(i)\}$ are distinct.

When $[n_i, n_j]/(n_i, n_j)$ is odd, the integ in (1) does not vanish; however, it may made arbitrarily small by proper ch of n_i and n_j. A set of functions wit small integral error may be termed qu orthogonal, or orthogonal to within so small error $(n_i, n_j)/[n_i, n_j]$. If the eleme of $\{n_i\}$ are distinct and relatively pri then the maximum integral error is eq to or less than $1/n_r n_s$, where n_r and are the two least odd elements of $\{$ When these numbers are large, the "se orthogonality" is established.

Fig. 1 gives the error coefficients l_{ij} a set $\{n_i = 1, 2, \cdots 18\}$. Notice in t figure how the numbers become, in gene more nearly orthogonal as n_i and n_j larger (that is, farther down the diagon

Although no two of the code symb have the same bandwidth, such a differe need not affect system performance i code-symbol set $\{n_i\}$ is chosen such t each n_i is large, and such that the ra n_i/n_j for each n_i and n_j in $\{n_i\}$ is appro mately unity.[4]

To any given set of semi-orthogo square waves with a specified orthogo error, other elements may be added with increasing the error by merely choos the n_a of such an added function $\phi_a(t)$ be a multiple of some n_k already in the $\{n_i\}$.

Proof of Theorem

Consider the square waves of n_i, half cycles, and let $v = (n_i, n_j)$. Divide interval $(t = 0, 1)$ into v subintervals the points $t = k/v$, $k = 1, 2, \cdots v -$ Only at these points do both functi cross the zero axis, and therefore, si each is a square wave, the product funct possesses even symmetry about these poi The product also has symmetry about mid points of the subintervals, because components are square. This latter sy metry is odd if $[n_i/v,\ n_j/v]$ is an e integral, since then either n_i/v or $n_j/$ even and the other odd; hence, one of functions $\phi_i(t)$, $\phi_j(t)$ goes through here, and the other is midway between crossings. The product thus crosses thro zero and is an odd function about midpoint. Similarly, when $[n_i/v,\ n_j/v]$ is odd integer, the product is an even funct about the midpoint. The numbers n_i/v n_j/v are relatively prime, and the inte in (1) is

$$\int_0^1 \phi_i(t)\phi_j(t)\,dt = \sum_{k=0}^{v-1} \int_{k/v}^{(k+1)/v} \phi_i(t)\phi_j(t)\,d$$

$$= v \int_0^{1/v} \phi_i(t)\phi_j(t)\,dt.$$

* Received by the PGIT, October 16, 1959.
[1] D. A. Bell, "Information Theory and its Engineering Applications" Pitman Publishing Co., New York, N. Y., second ed.; 1956.
[2] R. Price and P. E. Green, Jr., "A communications technique for multipath channels," Proc. IRE, vol., 46, pp. 555–569; March, 1958.

[3] R. W. Sanders, "Communication Efficiency of Space Telemetry Systems," *Proc. National Telemetry Conference*, Denver, Colo.; May, 1959.

[4] R. C. Titsworth and L. R. Welch, "Modul by Random and Pseudo-Random Sequences," Propulsion Lab., Pasadena, Calif., Prog. Rept 20-397; June 5, 1959.

I_{ij}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1		0	1/3	0	1/5	0	1/7	0	1/9		1/11	0	1/13	0	1/15	0	1/17	0
2	0		0	0	0	1/3	0	0	0	1/5	0	0	0	1/7	0	0	0	1/9
3	1/3	0		0	1/15	0	1/21	0	1/3	0	1/33	0	1/39	0	1/5	0	1/51	0
4	0	0	0		0	0	0	0	0	0	0	1/3	0	0	0	0	0	0
5	1/5	0	1/15			0	1/35	0	1/45	0	1/55	0	1/65	0	1/3	0	1/85	0
6	0	1/3	0	0	0		0	0	0	1/15	0	0	0	1/21	0	0	0	1/3
7	1/7	0	1/21	0	1/35	0		0	1/63	0	1/77	0	1/91	0	1/105	0	1/119	0
8	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0	0	0
9	1/9	0	1/3	0	1/45	0	1/63	0		0	1/99	0	1/117	0	1/15	0	1/153	0
10	0	1/5	0	0	0	1/15	0	0	0		0	0	0	1/35	0	0	0	1/45
11	1/11	0	1/33	0	1/55	0	1/77	0	1/99	0		0	1/143	0	1/165	0	1/187	0
12	0	0	0	1/3	0	0	0	0	0	0	0		0	0	0	0	0	0
13	1/13	0	1/39	0	1/65	0	1/91	0	1/117	0	1/143	0		0	1/195	0	1/221	0
14	0	1/7	0	0	0	1/21	0	0	0	1/35	0	0	0		0	0	0	1/63
15	1/15	0	1/5	0	1/3	0	1/105	0	1/15	0	1/165	0	1/195	0		0	1/255	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0
17	1/17	0	1/51	0	1/85	0	1/119	0	1/153	0	1/187	0	1/221	0	1/255	0		0
18	0	1/9	0	0	0	1/3	0	0	0	1/45	0	0	0	1/63	0	0	0	

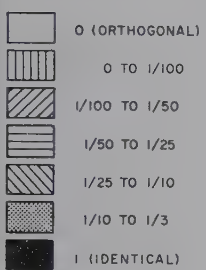

Fig. 1.

The integral is zero if $\phi_i(t)$, $\phi_j(t)$ is an d function about the midpoint; this urs when $[n_i/v,\ n_j/v]$ is even, and nishes the proof for the first part of the orem.

The integral of the product for even dpoint symmetry will now be calculated. $n_i/v = n$, $n_j/v = m$. These integers both odd, and relatively prime. Divide subinterval $(t = 0, 2/v)$ into $2mn$ equal ts; let $x = (vnm)t$ and

$$x) = \frac{1}{2n} \sum_{p=0}^{n-1} \sum_{q=0}^{2n-1} \omega_{2n}^{q(x-p)};$$

$$x = 0, 1, \cdots, 2mn - 1, \qquad (4)$$

ere ω_{2n} is a primitive $2n$th root of unity. is discrete function is unity at those nts where $\phi_i(t)$ is $+1$ in the corresponding erval, and zero where it is -1. Thus) may be represented by $[2S_i(x) - 1]$ discrete points $x = 0, 1, \cdots, 2mn - 1$. summation of the form

$$\sum_{q=0}^{2n-1} \omega_{2n}^{qx'} = \begin{cases} 1 & \text{if } x' \equiv 0 \bmod 2n; \\ 0 & \text{if } x' \not\equiv 0 \bmod 2n. \end{cases} \qquad (5)$$

ting $x' = x - p$, and summing from $= 0$ to $n - 1$, generates a square wave he proper duration and form.

The product has period $1/v$, and hence the integral from 0 to $2/v$ is twice the integral over 0 to $1/v$. The product in this former range may be represented as

$$\phi_i(t)\phi_j(t) = 1 - 2S_i(t) - 2S_j(x) + 4S_i(x)S_j(x) \qquad (6)$$

as shown in Fig. 1.

Such a representation is advantageous in that each nonzero point in the discrete function is proportional to the integral over the next adjacent interval of width $\Delta t = 1/nm$; hence, the integral over 0 to $1/v$ is

$$\int_0^{1/v} \phi_i(t)\phi_j(t)\,dt = \frac{1}{2vnm} \sum_{x=0}^{2nm-1} [1 - 2S_i(x) - 2S_j(x) \cdot + 4S_i(x)S_j(x)]. \qquad (7)$$

The first three terms of the integral summation are

$$\frac{1}{2nmv} \sum_{x=0}^{2nm-1} [1 - 2S_i(x) - 2S_j(x)] = \frac{1}{2nmv} [2nm - 2nm - 2nm] = -\frac{1}{v}. \qquad (8)$$

The last term is

$$\frac{1}{2n^2m^2v} \sum_{x=0}^{2nm-1} \sum_{q=0}^{2n-1} \sum_{p=0}^{m-1} \sum_{r=0}^{2m-1} \sum_{s=0}^{n-1} \cdot \omega_{2n}^{q(x-p)} \omega_{2m}^{r(x-s)} = \frac{1}{2n^2m^2v} \cdot \sum_{q,p,r,s} \omega_{2n}^{-qp} \omega_{2m}^{-rs} \sum_x [\omega_{2n}^q \omega_{2m}^r]^x. \qquad (9)$$

Choose the primitive $\omega_{2n} = e^{j\pi/n}$, then $\omega_{2m} = e^{j\pi/m}$. The inner term of (9) is a geometric series whose sum is

$$\sum_{x=0}^{2mn-1} \exp\left[+j2\pi\left(\frac{mq + nr}{2mn}\right)x\right] = \frac{1 - e^{j2\pi(mq+nr)}}{1 - \exp\left[j2\pi\,\dfrac{(mq + nr)}{2nm}\right]}. \qquad (10)$$

Clearly, the right side of (10) vanishes, unless $mq + nr/2nm$ is integral; if this is the case, then

$$mq + nr = 2nmk$$

for some integer k. Arrange this equation as

$$m(2nk - q) = nr. \qquad (11)$$

One solution is $q = r = 0$. If q and r are nonzero, then m must divide r, and n must divide q, since $(n, m) = 1$. In the range of q and r, the only solution is $q = n$, $r = m$. Under these circumstances, the sum is a sum of $2mn$ ones.

Thus, the sum in (9) is

$$\frac{1}{vnm} \sum_{p,s} [1 + (\omega_{2n}^{-n})^p(\omega_{2m}^{-m})^s] = \frac{1}{vnm} \sum_{p=0}^{m-1} \sum_{s=0}^{n-1} 1 + \frac{1}{vnm} \sum_{p=0}^{m-1} (-1)^p \sum_{s=0}^{n-1} (-1)^s. \qquad (12)$$

Since m, n are odd, then this final term is $1/v + 1/vnm$. The entire expression in (7) is

$$\int_0^{1/v} \phi_i(t)\phi_j(t)\,dt = -\frac{1}{v} + \frac{1}{v} + \frac{1}{vnm} = \frac{1}{vnm}. \qquad (13)$$

The product nm is $n_in_j/v^2 = [n_i, n_j]/v$; thus, by (3) the total integral is v times the subintegral given in (13),

$$\int_0^1 \phi_i(t)\phi_j(t)\,dt = \frac{1}{nm} = \frac{(n_i, n_j)}{[n_i, n_j]} \qquad (14)$$

and the theorem is complete.

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Solution of an Integral Equation Occurring in Multipath Communication Problems*

In some recent work on multipath communication we obtained an integral equation similar to one discussed by D. Middleton.[1] This equation is further discussed in an erratum by D. Middleton in this issue.[2] We shall describe briefly some other methods of solution which in some cases give more "practical" answers. The equation to be solved is

$$\rho(t, \tau) = \int_0^T h(x, t)h(x, \tau)\, dx, \qquad 0 \leq t, \tau \leq T, \tag{1}$$

where $\rho(t, \tau)$ is a known symmetrical positive-definite function, and the $h(x, t)$ is unknown and is to be determined. Since x is limited by the range of integration to the interval $(0, T)$ it can readily be shown by application of the frequency sampling theorem[3,4] that we may write $h(x, t)$ in the form

$$h(x, t) = \sum_{n=-\infty}^{\infty} f_n(t) \exp\left(\frac{j2\pi nx}{T}\right), \qquad 0 \leq x, t \leq T. \tag{2}$$

Substituting from (2) into (1) we get

$$\begin{aligned}\rho(t, \tau) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_n(t)f_m(\tau) \cdot \int_0^T \exp\left(\frac{j2\pi(n+m)x}{T}\right) dx \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_n(t)f_n(\tau) \cdot T \cdot \delta(n+m) \\ &= \sum_{n=-\infty}^{\infty} Tf_n(t)\overline{f_n(\tau)} \\ &= Tf_0(t)\overline{f_0(\tau)} + 2\sum_{n=1}^{\infty} Tf_n(t)\overline{f_n(\tau)},\end{aligned} \tag{3}$$

where the bar denotes the complex conjugate function.

Now since $\rho(t, \tau)$ is real, symmetrical, and positive-definite it can be expanded by Mercer's theorem[5] in the form

$$\rho(t, \tau) = \sum_{n=1}^{\infty} \lambda_n \phi_n(t)\overline{\phi_n(\tau)}, \qquad 0 \leq t, \tau \leq T, \tag{4}$$

where

$$\lambda_n \phi_n(t) = \int_0^T \rho(t, \tau)\phi_n(\tau)\, d\tau \tag{5}$$

and the λ_n are real and nonnegative. From (3) and (4) we see that the $f_n(t)$ can be obtained by suitable identification with the $\varphi_m(t)$. It is clear that this identification is not unique, but can be made in many ways. Furthermore the constant T in (3) may be written $\sqrt{T}\exp(j\varphi)\cdot\sqrt{T}\exp(-j\varphi)$ where φ is arbitrary, so that we can obtain an infinity of solutions with arbitrary phase functions for the $f_n(t)$. When a choice for the $f_n(t)$ has been made, the solution for $h(x, t)$ follows from (2). We may use our great freedom of choice to obtain the simplest form for $h(x, t)$ and perhaps also to avoid any divergent solutions. This matter of convergence is not easily discussed in general terms but can be settled for each particular case. Notice too that in our derivation we have not restricted $h(x, t)$ in any way outside the square $(0 \leq x, t \leq T)$. This may also be of advantage in some situations. We may also notice that we could obtain a similar solution by using for $h(x, t)$ any other expansion, $\sum_n f_n(t)g_n(x)$, for instance, where the $g_n(x)$ are orthogonal over T, instead of the trigonometric expansion used in (2). The $g_n(t)$ may then be chosen to fit particular situations.

Mercer's theorem suggests another form of solution for $h(x, t)$; *viz.*,

$$h(x, t) = \sum_{n=1}^{\infty} (\lambda_n)^{1/2}\phi_n(t)\phi_n(x), \qquad 0 \leq x, t \leq T, \tag{6}$$

where $(\lambda_n)^{1/2}$ is real because the λ_n are real and positive, although some of them might be zero. This solution, of course, is the analog in the continuous case of the well-known method for proving that a real symmetric positive-definite matrix A can be written as the product of a real nonsingular matrix and its transpose.[6]

In general, the expansion for $\rho(t, \tau)$ by Mercer's theorem will have an infinite number of terms unless (and in fact, only if) $\rho(t, \tau)$ is a "separable" kernel; that is, $\rho(t, \tau)$ can be written

$$\rho(t, \tau) = \sum_{n=1}^{N} u_n(t)v_n(\tau). \tag{7}$$

If the $u_n(t)$ and $v_n(t)$ are exponential functions, the eigenfunctions of $\rho(t, \tau)$ (5), are readily found and are, in fact, themselves linear combinations of exponential functions. Such a $\rho(t, \tau)$ will give solutions by the first method for $h(x, t)$ of the type

$$h(x, t) = \text{sum of terms like } e^{-\alpha t}\cos\beta x, \qquad 0 \leq x, t \leq T. \tag{8}$$

Thus, for kernels of the separable t[y] this method often gives a conven[ient] answer. In other cases it may sometime[s] possible to sum the infinite series for $h($[…] or to approximate it suitably by a fi[nite] sum. In applications, it is often requ[ired] to realise $h(x, t)$ of (2), (6), (8) as ti[me-]variant filters. Some methods of realis[ing] time-variant filters of this type were gi[ven] previously.[4] Notice that both methods [of] solution give filters $h(x, t)$ that are [in] general unrealizable, though they [can] always be made realisable with a dela[y of] not more than T seconds.

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[1] D. Middleton, "On the detection of stochastic signals in additive normal noise—part I," IRE TRANS. ON INFORMATION THEORY, vol. IT-3, pp. 86–121, eqs. (277)–(285); June, 1957.

[2] D. Middleton, this issue, p. 349.

[3] P. M. Woodward, "Probability and Information Theory with Applications to Radar," Pergamon Press, London, Eng.; 1953.

[4] T. Kailath, "Sampling Models for Linear Time-Variant Filters," Res. Lab. of Electronics, Mass. Inst. Tech., Cambridge, Mass., Tech. Rept. no. 352; May, 1959.

[5] W. B. Davenport and W. L. Root, "An Introduction to Random Signals and Noise," McGraw-Hill Book Co., Inc., New York, N. Y.; 1958.

[6] E. A. Guillemin, "The Mathematics of Circuit Analysis," John Wiley & Sons, Inc., New York, N. Y.; 1949.

Correction to "On the Detection [...] of Stochastic Signals in Additive [...] Normal Noise-Part I"

The interpretation of optimum sys[tem] structure for the detection of normal n[oise] signals in additive normal noise descri[bed] in the previous paper[1] is not entirely cor[rect] in the case of the time-invariant filter sho[wn] in Fig. 2 therein. It is not possible to h[ave] a time-invariant filter alone here, if fi[nite] samples are considered; a time-vary[ing] switch is needed, in addition. In all ca[ses] the linear element preceding the z[ero-]memory square-law device (or multip[lier] in the second interpretation) is ti[me-]varying (and may or may not be real[iz]able). Accordingly, in (77), (131), ([…]) the upper limit on the integral should [be] t, rather than T, while to write (276) imp[lies] the aforementioned time-varying swi[tch]. Fig. 2 should be modified accordin[gly] also. Moreover, it is also not possibl[e to] choose a purely invariant linear filter [in] (275) for suboptimum systems, with […] similarly implying a time-varying swi[tch]. For a full discussion of the optimum fi[lter] here and in general threshold deci[sion] systems, see the author's paper in [the] present issue.[2] The author is indebte[d to] T. Kailath[3] for calling attention to [the] incomplete interpretation of structur[e in] the present instance.

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[1] D. Middleton, "On the detection of stoc[hastic] signals in additive normal noise-part I," IRE T[RANS.] ON INFORMATION THEORY, vol. IT-3, pp. 86[–121;] June, 1957.

[2] D. Middleton, "On New Classes of Ma[tched] Filters and Generalizations of the Matched [Filter] Concept," IRE TRANS. ON INFORMATION TH[EORY,] this issue, pp. 349–360.

[3] For a discussion of related time-varying f[ilters] and some further references, see T. Kailath, "So[lution] of an integral equation occurring in mul[tipath] communication problems," this issue, pp. 41[2].

Contributors

Phillip Bello (S '52—A '55) was born on ctober 22, 1929 in Lynn, Mass. He re- eived the B.S.E.E. degree from Northeastern University, Boston, Mass., in 1953 and the M.S. and D.Sc. degrees from the Massachusetts Institute of Technology, Cambridge, in 1955 and 1959, respectively.

P. Bello

From 1955 to 1957, he was first a research associate and then an assistant professor of com- unications at Northeastern University, eaching courses in the area of linear ystem analysis and statistical communi- ation theory. During this time, he was lso involved in certain classified research the latter field.

From 1956 to 1958, he was employed by unn Engineering Associates, Cambridge, here he was engaged in analytical studies ssociated with various aspects of statisti- al radar theory. Since 1958, he has been nployed at the Applied Research Labo- atory of Sylvania Electronic Systems, Valtham, Mass.

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❖

Louis J. Cutrona (M '46—SM '47) was orn March 11, 1915, in Buffalo, N. Y. Ie received the B.A. degree in physics from Cornell University, Ithaca, N. Y., in 1936 and attended the University of Buffalo, Buffalo, N. Y., from 1936 to 1937. He received the M.A. and Ph.D. degrees in physics from the University of Illinois, Urbana, in 1938 and 1940 respectively. From 1940 to 1942, he was n instructor of physics and electrical engi- eering at the Duluth Junior College, Duluth, Minn. From 1942 to 1943, he was member of the staff of the York Safe and ock Company, and he was a member of he technical staff of Bell Telephone aboratories, from 1943 to 1945. From 945 to 1947, he was an employee of the ederal Telecommunications Corporation, om 1947 to 1949, of the Sperry Gyroscope ompany; and in 1949, he joined the Willow un Laboratories of The University of lichigan, Ann Arbor. He has been asso- ated with the University of Michigan since that time, except for the year 1956–1957 when he was on leave to the Space Technology Laboratory, Los Angeles, Calif. In 1957, he received an appointment as professor of electrical engineering at The University of Michigan. For the year 1959–1960, he is on leave from that position as head of the Radar Laboratory of the Willow Run Laboratories. His work has been in a variety of fields, particularly those of circuit theory, information theory, communications, radar, and missile system design.

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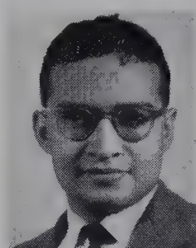
Thomas Kailath (S '57) was born on June 7, 1935, in Poona, India. He received the B.E. degree in telecommunications engineering in 1956 from the College of Engineering, Poona.



T. Kailath

From 1956 to 1957, he was an assistant lecturer in electrical engineering at the L.D. College of Engineering, Ahmedabad, India. In 1957, he came to the Massachusetts Institute of Technology, Cambridge, as a graduate student in electrical engineering and research assistant in the Information Theory Group of the Research Laboratory of Electronics. He received the M.S. degree from M.I.T. in 1959, and is currently working toward the D.Sc. degree. During the summer of 1959, he was employed by the Lincoln Laboratory, M.I.T., Lexington, Mass., doing research in communication theory.

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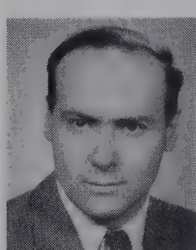
Emmett N. Leith was born in Detroit, Mich., on March 12, 1927. He received the B.S. degree in physics in 1950 from Wayne State University, Detroit, Mich., and the M.S. degree in physics in 1952 from the same institution. He joined the Willow Run Laboratories of the University of Michigan, Ann Arbor, in 1952, where he is presently employed.



E. N. Leith

His work has been in the areas of radar, microwaves, communication theory, data processing, and optics.

Mr Leith is a member of Sigma Pi Sigma and Sigma Xi.

Robert M. Lerner (SM '57) was born on February 1, 1928, in Worcester, Mass. He received the B.S. and M.S. degrees in 1948 and 1950, respectively, from Worcester Polytechnic Institute, Worcester, Mass.

R. M. Lerner

Subsequently, he held a Swope fellowship at Massachusetts Institute of Technology, Cambridge, and in 1951, joined the Research Laboratory of Electronics to work on speech processing. In 1952, he joined the staff of Lincoln Laboratory, Lexington, Mass., to work on the application of statistical techniques to the design of radar systems. Since early 1957, he has been Assistant Group Leader of a Communication Systems Group at Lincoln Laboratory. He is now primarily concerned with the application of theoretical techniques to communications system design, and with the development of acoustic components. He received the D.Sc. degree from M.I.T. in 1959.

Dr. Lerner is a member of the Acoustical Society of America, Sigma Xi, and Tau Beta Pi.

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David Middleton (S '42—A '44—M '45—SM '53—F '59) was born on April 19, 1920, in New York, N. Y. He received the A.B. degree in 1942 the M.A. degree in 1945, and the Ph.D. degree in physics in 1947, all from Harvard University, Cambridge, Mass.

D. Middleton

During World War II, he was a research associate at the Harvard Radio Research Laboratory, working in the field of electronic countermeasures. From 1947 to 1949, he was a Research Fellow in electronics at the Harvard Electronics Research Laboratory of the Division of Applied Science. In 1949, he became assistant professor of applied physics in the same division. Since 1954, he has engaged in private consulting practice with industry and the armed services. At present, his principal field of research is in statistical communication theory, including applications in electronics, electron physics, information theory, system design and evaluation, and the study of various problems in applied mathematics related to these fields.

Dr. Middleton is a Fellow of the Ameri-

can Physical Society and of the AAAS, a member of the American Mathematical Society, New York Academy of Sciences, Society for Industrial and Applied Mathematics, Institute of Mathematical Statistics, Phi Beta Kappa, and Sigma Xi. He was a National Research Council predoctoral fellow in physics from 1946 to 1947, and received the National Electronics Conference Award (with W. H. Huggins), in 1956. His book "An Introduction to Statistical Communication Theory," International Series in Pure and Applied Physics, McGraw-Hill Book Co., Inc., appeared in April, 1960.

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Carmen J. Palermo was born in Rochester, N. Y., on July 17, 1933. He received the B.S. degree in electrical engineering from Cornell University, Ithaca, N. Y., in 1956. Since then he has been a member of the Radar Laboratory of Willow Run Laboratories, Ypsilanti, Mich. During this time he received the M.S. degree in physics from the University of Michigan, Ann Arbor. At present he is with the analytical group, working in the area of signal processing.


C. J. Palermo

Mr Palermo is a member of Eta Kappa Nu and Sigma Xi.

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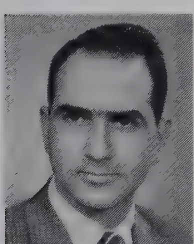
Leonard J. Porcello (S '57—M '59) was born on March 1, 1934, in New York, N. Y. He received the B.A. degree in physics from Cornell University, Ithaca, N. Y., in 1955, the M.S. degree in physics in 1957, and the M.S.E. degree in electrical engineering in 1959, both from The University of Michigan, Ann Arbor.


L. J. Porcello

In 1955 he was appointed as a research assistant at the Willow Run Laboratories of the University of Michigan, and in 1958 was appointed as an instructor in the Electrical Engineering Department of the University of Michigan, which position he holds at present. He has continued to work on problems in the areas of optical data processing and filtering schemes, as well as on problems attendant to research and development in the area of radar systems.

Mr Porcello is a member of Sigma Xi.

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Roger H. Prager (S '55—M '56) was born in New York, N. Y., on November 9, 1931. He received the B.A. degree from the University of Chicago, Ill., in 1951, and the B.S.E.E. degree from the Massachusetts Institute of Technology, Cambridge, in 1955. He spent a year and a half as a member of the research staff in the M.I.T. Acoustics Laboratory, engaged in research on the statistical analysis of sonar signals using analog computer techniques.


R. H. Prager

Since January, 1957, Mr. Prager has been an electronics scientist at the U. S. Navy Electronics Laboratory in San Diego, Calif. His work has been in the area of sonar signal processing and display.

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James L. Stewart was born on January 5, 1918, in Szechwan, China, and became a U. S. citizen in 1949. He received the B.A. degree in 1938 and the M.A. degree in 1940 from the University of Saskatchewan, Saskatoon, Can., working in slow neutrons. He received the Ph.D. degree in 1943 from The Johns Hopkins University, Baltimore, Md., with a dissertation in ultrasonics in gases which led to the discovery of rotational relaxation in hydrogen.


J. L. Stewart

He served with CARDE, Valcartier, Quebec, Can., from 1943 to 1945, working on recoil ballistics, and was an assistant professor of physics at Rutgers University, New Brunswick, N. J., from 1946 to 1951. He joined the U. S. Navy Electronics Laboratory, San Diego, Calif., in 1951, specializing in the application of correlation analysis and techniques to underwater acoustics.

Dr. Stewart is an Acoustical Society of America Fellow and a member of Sigma Xi.

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Steven M. Sussman was born in Pforzheim, Germany on August 7, 1929. He received the B.S. and M.S. degrees in electrical engineering at the Massachusetts Institute of Technology, Cambridge, in 1953.


S. M. Sussman

He joined the Applied Science Division of Melpar Inc., Falls Church, Va., in 1954 where he has since been engaged in studies on communication and detection systems for the milita services.

Mr. Sussman is a member of Sigma N Tau Beta Pi, and Eta Kappa Nu.

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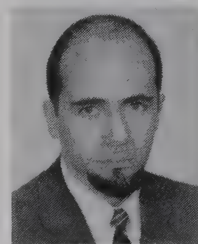
George L. Turin (M '56—SM '59) w born in New York, N. Y., on January 2 1930. He received the S.B. and S.M. degre from the Massach setts Institute Technology, Car bridge, Mass. in 19 after completing t cooperative course electrical engineeri in association wi Philco Corporatio In the summer 1952 he was M.I.T. Overseas Fe low at Marcon Wireless Telegrap Co. in England. From 1952–1956 he worke at M.I.T.'s Lincoln Laboratory, Lexingto Mass., in the field of statistical communic tion theory, first as a staff member, a later as a research assistant while completi his doctoral studies. During this latt period he was also a consultant to the fir of Edgerton, Germeshausen, and Grie He received the Sc.D. degree in electric engineering from M.I.T. in 1956.


G. L. Turin

Since July, 1956 Dr. Turin has bee engaged in communication and rad research studies at Hughes Aircraft Co pany, Culver City, Calif. He has also taug part-time at the University of Southe California, Los Angeles, and the Universi of California at Los Angeles.

Dr. Turin is a member of Eta Kappa N Tau Beta Pi, and Sigma Xi. He is also member of Commission 6.1 and 6.2 of t U. S. National Committee of URSI.

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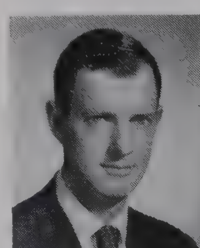
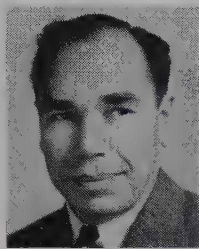
George R. Welti (M '56) was born Muri, Switzerland, on April 23, 1923. H received the S.B. and S.M. degrees mechanical eng neering and the S.M degree in electric engineering from t Massachusetts Ins tute of Technolog Cambridge, in 194 1948, and 1955, spectively.


G. R. Welti

From 1948 to 19 he was employed a project engine at the Sperry Gyr scope Compan Great Neck, N. Y., where he worked stable platforms and fire control compute From 1951 to 1956 he served as divisi manager at M.I.T.'s Dynamic Analy and Control Laboratory, where he was sponsible for development of analog co puter systems and components, and whe he performed machine analyses of int

eption and terminal control problems. In 956, Mr. Welti joined Hycon Eastern, nc., Cambridge, Mass., as a staff member, where he studied redundant automata and nformation coding. Since 1956, he has been mployed by the Raytheon Co., Waltham, Mass., as a staff engineer, where he has erformed analyses of radar, navigation, ommunication, and anti-jamming systems. He is presently serving as manager of he Advanced Development and Systems Analysis Department of the Communiations and Data Processing Operations.

Mr. Welti is a member of Pi Tau Sigma nd Sigma Xi.


E. C. WESTERFIELD

Everett C. Westerfield (M '56) was born in Beda, Ky., on April 3, 1901. He received the B.A. degree in physics in 1928, the M.A. degree in mathematics in 1930, and the Ph.D. degree in physics in 1940, all from the University of Colorado, Boulder.

From 1941 to 1945, he was employed at Douglas Aircraft Company, Long Beach, Calif., as a process control engineer in charge of physical testing of aircraft material and parts. From 1945 to 1946, he was an associate physicist at the University of California Division of War Research, San Diego, studying the scattering of underwater sound. Since 1946, he has been employed as a civilian scientist at the U. S. Navy Electronics Laboratory, San Diego, Calif., doing research and theoretical studies in sonar reverberation, communication and detection problems.

Dr. Westerfield is a member of the American Physical Society and the American Mathematical Society.

Abstracts Section

This section of the TRANSACTIONS describes material which may be of interest to PGIT members and which is not readily available in United States professional journals. Industrial and University reports are sources for much of the material; also, articles originating outside the United States are emphasized. It is suggested that members familiar with such material communicate the pertinent information to the Abstract Editors: Richard A. Epstein, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif.; George L. Turin, Hughes Aircraft Company, Culver City, Calif.

Sequences with Randomness Properties, by S. W. Golomb, (The Martin Co.); June, 1955.

Linear recurring binary sequences and their associated polynomials are investigated. The necessary condition for maximum-length sequences is that the associated polynomial be irreducible. Exact formulas for the number of irreducible polynomials and for the number corresponding to the maximum sequence period $P = 2^n - 1$ are given. The "randomness" conditions for maximum-length linear recurring sequence (MLLRS) are specified as:

1) In each period there are $(P + 1)/2$ ONEs and $(P - 1)/2$ ZEROs.
2) There are 2^{n-k-2} runs of k consecutive ONEs (as well as of ZEROs) in each period, for $1 \leq k \leq n - 2$, plus one run of $n - 1$ ZEROs and one run of n ONEs.
3) The periodic autocorrelation function $C(\tau) = 1/P \sum^{P}_{k=1} a_k a_{k+\tau}$ takes on only two values: $C(0) = (P + 1)/2P$, $C(\tau \neq 0) = (P + 1)/4P$.

The correlation property is particularly useful in recovering phase (or range) information from a signal in a noisy environment.

The MLLRS sequences are characterized by the "cycle-and-add" property:

$$\{a_k\} + \{a_{k+\tau}\} = \{a_{k+\tau'}\}.$$

They are all obtainable as linear shift register sequences, a significant mechanization advantage over other sequences with two-level correlation.

The final chapter discusses computational methods for finding the polynomials which correspond to the MLLRS.

Nonlinear Shift-Register Sequences, by S. W. Golomb and L. R. Welch, (Jet Propulsion Lab., Cal. Inst. Tech.); Memo. No. 20–149; October 25, 1957.

Results of recent investigations into the mathematical theory of binary nonlinear shift-register sequences are summarized. These are sequences derivable from recirculating shift register in which the combining operations are allowed to be of forms other than logical addition. The correlation and periodicity properties of these sequences are discussed and the work of de Bruijn on maximum-length sequences is extended. A statistical model is suggested for the distribution of cycle lengths. Tabulations of values obtained in extensive experimental research are also included together with a discussion of their significance.

Pulse Compression, By R. Kronert, *Nachrichtentech.*; Vol. 7, April, 1957. (Translation available from Crerar Library, Chicago, Ill.)

A method is described that enables frequency modulated HF pulses to be "compressed"—*i.e.*, decreased in width and increased in amplitude. A high frequency Gaussian impulse is considered wherein the carrier frequency rises linearly with time; the compression after transmission through a filter with phase delay distortion is computed, an ideal filter for the purpose of compression being assumed. An optimum transit-time gradient (*i.e.*, an optimum number of sections) is thus obtained for a given impulse shape. The extent to which the behavior of all-pass filters approximates the case of ideal compression is then examined.

In Part II, of Kronert in the July, 1957 issue of *Nachrichtentechnik*, experimental demonstrations of the theoretical results are described.

Experimental Study of Tapped Delay-Line Filters, by D. W. Lytle, (Stanford Electronics Labs.) Tech. Rept. No. 361-3; July 30, 1956.

A method of synthesizing a matched filter is developed through the use of the sampling theorem. The matched filter, and also the matched signal generator is composed of tapped ideal delay lines, low-pass filters, and adding circuits. In theory, use of the method suggested allows for any finite bandwidth and finite time duration signal as the matched filter. Binary signals whose matched-filter outputs are close to ideal (pseudorandom signals) are considered. A measure of the quality of these signals is defined so that different metched signals may be compared. Results of experiments on a tapped delay line matched filter are reported.

On the Properties of Matched Filters, by D. W. Lytle, (Stanford Electronics Labs.); Tech. Rept. No. 17, June 10, 1957.

Matched filters are defined and their properties as optimum signal detectors discussed. Conditions under which matched filters are realizable are enumerated. Thes conditions are not unduly rsetrictive an a variety of synthesis methods are appl cable. The method providing greates flexibility and adaptability is the synthes procedure which employs tapped dela lines. This mode of synthesis is base upon the sampling theorem of frequenc limited time functions.

Two novel methods of synthesizin matched filters are investigated. Thes types, termed multiple band-pass and al pass inverse-matched filters, may b employed when the matched signal prescribed. The latter type is shown to b inferior to the tapped-delay-line matche filter because of its low effective-time duration matched signal. The multipl band-pass inverse-matched filter may hav a large effective-time-duration matche matched signal, but it does not presen the flexibility or ease of synthesis offere by tapped delay lines.

Range and Velocity Accuracy from Rada Measurements, by R. Manasse, (M. I. T Lincoln Lab.); Group Rept. 312–26; Febru ary 3, 1955.

A method for the determination of rang and Doppler velocity with a radar syster in an optimum manner with the use o matched filters as well as the accurac obtainable with this procedure are discusse The methods of Woodward are utilized t derive the noise moment matrix for th range and velocity estimates and, conse quently, the variances of their distributio Two appendixes are included that conside the cases of a pulse of sine wave and pulse-modulated sine wave.

An Analysis of Angular Accuracies fro Radar Measurements, by R. Manass (M. I. T. Lincoln Lab.); Group Rept. 32–2 December 6, 1955.

A theoretical formula for the angul accuracy of a radar measurement is derive For reasonable signal detectability, it shown that this angular accuracy is a sma fraction of the antenna beamwidt Optimum detection procedure requires large number of separate receivers process the received energy; however, a approximate method of processing receive angular information which compares th

put from only two receivers, as in a nopulse technique, results in an accuracy ich compares favorably with the imum.

he Application of the Theory of Signal tectability to Signals with Unknown arization and Phase, by R. Manasse, . I. T. Lincoln Lab.); Group Rept. 32–25; gust 20, 1956.

he general theory of signal detectability eviewed and, in particular, is summarized signals in additive white noise which are own exactly except for phase. The theory then extended to include signals with own polarization. The optimum receiver, ikelihood ratio detector, may be imple- nted by an appropriate processing of tputs from linear filters matched to hogonal components of signal polariza- n. Detection of signals with unknown ase and polarization is studied in relation to the radar problem. A conclusion is that the total effecrive target cross section is basically more significant in determining the detectability than is the effective cross section corresponding to a particular received polarization. Detectability curves and tables are presented.

Studies of Woodward's Uncertainty Function, by W. M. Siebert, (M. I. T. Res. Lab. of Electronics); Quart, Prog. Rept. April 15, 1958.

A summary of known properties of certain functions significant to the theory of radar synthesis is presented. These functions, the so-called θ and ψ functions of Ville, Woodward, and Siebert, involve Fourier transforms of products of complex time-functions. Several theorems are given concerning representations, coordinate transformation, and necessary and /or sufficient conditions for such functions. Certain unsolved problems concerning these functions are indicated.

On Raising the Sensitivity of Interference Radio Telescopes, II, by G. M. Tovmasian; *Dokl. Armenian SSR*, vol. 26; 1958. (Translation available from Crerar Library, Chicago, Ill.)

This paper is Part II of an investigation into the possibilities of raising the sensitivity of phase-switched interferometer radiotelescopes by passing the output into a resonant oscillating system having a natural frequency equal to the frequency with which a point source of radio emmission passes through the lobes of the interference pattern. This method results in an increase of sensitivity, an increase in declination resolving power, and in certain favorable cases, an increase in the right ascension resolving power.

INFORMATION FOR AUTHORS

Authors are requested to submit editorial correspondence or technical manuscripts to the Publications Chairman for possible publication in the PGIT Transactions. Papers submitted should include a statement as to whether the material has been copyrighted, previously published, or accepted for publication elsewhere.

Papers should be written concisely, keeping to a minimum all introductory and historical material. It is seldom necessary to reproduce in their entirety previously published derivations, where a statement of results, with adequate references, will suffice.

To expedite reviewing procedures, it is requested that authors submit the original and two legible copies of all written and illustrative material. The manuscript should be double-spaced, and the illustrations drawn in India ink on drawing paper or drafting cloth. Each paper should include a carefully written abstract of not more than 200 words. Upon acceptance, papers should be prepared for publication in a manner similar to those intended for the Proceedings of the IRE. Further instructions may be obtained from the Publications Chairman. Material not accepted for publication will be returned.

IRE Transactions on Information Theory is published four times a year, in March, June, September, and December. A minimum of one month must be allowed for review and correction of all accepted manuscripts. In addition, a period of approximately two months is required for the mechanical phases of publication and printing. Therefore, all manuscripts must be submitted three months prior to the respective publication dates.

All technical manuscripts and editorial correspondence should be addressed to George A. Deschamps, University of Illinois, Urbana, Ill. Local Chapter activities and announcements, as well as other nontechnical news items, should be addressed to Laurin G. Fischer, ITT Laboratories, 492 River Road, Nutley 10, N. J.

R.G. STEPHENSON

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